Learning Seminar on Deligne's Weil *II* Theorem

Organized by Bhargav Bhatt*

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1 May 3—Overview (Bhargav Bhatt)

The goal of the seminar is to prove the Riemann hypothesis part of the Weil conjectures. Today, we will formulate the statements and talk about how they can be interpreted using étale cohomology.

1.1 Weil conjectures

We will use notation from Deligne and our reference [KW01].

Notation 1.1. X_0 will denote a variety over $\mathbf{F}_q$, and $X \coloneqq X_0 \otimes_{\mathbf{F}_q} \overline{\mathbf{F}}_q$ will denote the corresponding variety over the algebraic closure of $\mathbf{F}_q$.

The idea is we want to compute how many $\mathbf{F}_{q^r}$-points there are on X_0. We do this by putting everything in a formal power series:

Definition 1.2. The *zeta function* for X_0 is defined as the formal power series

$$Z(X_0, t) = \exp\left(\sum_{r=1}^{\infty} \#X_0(\mathbf{F}_{q^r}) \frac{t^r}{r}\right) \in \mathbf{Q}[[t]].$$

Example 1.3. Let $X_0 = \operatorname{Spec}(\mathbf{F}_q)$. We get

$$Z(X_0, t) = \exp\left(\sum_{r=1}^{\infty} 1 \cdot \frac{t^r}{r}\right) = \frac{1}{(1-t)}.$$

Example 1.4. Let $X_0 = \mathbf{P}^1$. Then, $\#X_0(\mathbf{F}_{q^r}) = 1 + q^r$ by using the decomposition $\mathbf{P}^1 = \mathbf{A}^1 \cup \{\infty\}$, and so

$$Z(X_0, t) = \exp\left(\sum_{r=1}^{\infty} (1 + q^r) \frac{t^r}{r}\right) = \frac{1}{(1-t)(1-qt)}.$$

These are both rational functions: this is the first part of the Weil conjectures. Motivated by slightly more complicated examples (Fermat hypersurfaces), Weil formulated the following conjectures:

Conjectures 1.5 (Weil)**.** Let X_0 be a smooth projective variety over $\mathbf{F}_q$, which is geometrically connected of dimension n, and let $Z(t) \coloneqq Z(X_0, t)$.

1. *Rationality:* $Z(t)$ is a rational function.
2. *Functional equation:* $Z(1/q^n t)$ and $Z(t)$ are related up to a "fudge factor":

   $$Z\left(\frac{1}{q^n t}\right) = \pm q^{nE/2} t^E \cdot Z(t),$$

   where E is a number coming from the geometry of X (or X_0): $E = \Delta \cdot \Delta$, where $\Delta \subset X_0 \times X_0$ is the diagonal. [Alternatively, $E = c_{\text{top}}(T_{X_0})$ is the top Chern class of the tangent bundle, or $E = \chi_{\text{top}}(X)$ is the topological Euler characteristic (this uses 4).]
3. *Riemann hypothesis:* The rational function $Z(t)$ has a special form:

   $$Z(t) = \frac{P_1(t) P_3(t) \cdots P_{2n-1}(t)}{P_0(t) P_2(t) \cdots P_{2n}(t)},$$

   where each $P_i(t)$ satisfies the following properties:
   (a) $P_0(t) = 1 - t \in \mathbf{Z}[t]$;
   (b) $P_{2n}(t) = 1 - q^n t \in \mathbf{Z}[t]$;
   (c) For $1 \leq i \leq 2n-1$, we have

   $$P_i(t) = \prod_j (1 - \alpha_{ij} t) \in \mathbf{Z}[t],$$

   where each α_{ij} is an algebraic integer, and $|\alpha_{ij}| = q^{i/2}$, where $|\cdot|$ denotes the complex norm for any embedding of $\mathbf{Z}[\alpha_{ij}]$ in $\mathbf{C}$.

4. *Betti numbers:* If X_0 lifts to some Y in characteristic zero (i.e., its mod p reduction is X_0), then $\deg P_i(t) = \beta_i(Y \otimes \mathbf{C})$, the ith (topological) Betti number of $Y \otimes \mathbf{C}$.

We will assume 1 and 2 to be known (they are covered in introductory courses on étale cohomology), and focus on 3. We can see the Weil conjectures hold by inspection for the two examples above. In particular, for 4, you can see how the singular cohomology of $\mathbf{P}^1$ shows up in the zeta function for $\mathbf{P}^1$.

For particular examples, you can often see how to write down the lifting for 4 (Weil probably did this for hypersurfaces).

1.2 Review of étale cohomology

We now recall Grothendieck's formalism for tackling these conjectures.

1.2.1 Fundamental groups

To begin, to define an algebraic analogue of singular cohomology, we look at fundamental groups (note this is not how it happened historically).

Let X/k be a geometrically connected variety, and fix a base point $x\colon \operatorname{Spec}(\overline{k}) \to X$.

1. There exists a canonical profinite group $\pi_1(X, x)$ (independent in x, up to conjugation) such that it satisfies the following universal property:

$$\begin{pmatrix} \{\text{finite } \pi_1(X,x)\text{-sets}\} \\ + \text{ the forgetful functor} \end{pmatrix} \cong \begin{pmatrix} \{\text{finite étale covers } Y \to X\} \\ + \text{ fibre over } x \end{pmatrix}$$

The construction of such a group is in SGA1.

Example 1.6.
(a) $X = \operatorname{Spec} k$. Then, $\pi_1(X, x) \cong G_k = \operatorname{Gal}(\overline{k}/k)$
(b) $X = \mathbf{P}^1$. By Riemann–Hurwitz, there are no non-trivial finite covers of $\mathbf{P}^1_{\overline{k}}$, and so $\pi_1(X, x) \cong G_k$.

In this way, you can see that the fundamental group generalizes the Galois group to schemes.

2. If X/k is geometrically connected, with base point x, then

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X \otimes \overline{k}, x) \arrow[r] \arrow[d, phantom, "\rotatebox{-90}{$=:$}"] & \pi_1(X, x) \arrow[r] \arrow[d, phantom, "\rotatebox{-90}{$=:$}"] & \pi_1(\operatorname{Spec}(k)) \arrow[r] \arrow[d, phantom, "\rotatebox{-90}{$=:$}"] & 1 \\
& \pi_1^{\text{geom}}(X) & \pi_1^{\text{arith}}(X) & G_k &
\end{tikzcd}$$

where the notation in the second line is due to Katz.

1.2.2 Local systems

Now that we have fundamental groups, we can look at their representations as we do in topology to get local systems.

Fix a prime ℓ invertible in k. We work with $\overline{\mathbf{Q}}_\ell$ coefficients. We then get a category $\mathsf{Loc}(X, \overline{\mathbf{Q}}_\ell) =: \mathsf{Loc}(X)$.

Fact 1.7. Assume X is normal. Then, $\mathsf{Loc}(X) \cong \mathsf{Rep}^{\text{cont}}_{\overline{\mathbf{Q}}_\ell}(\pi_1(X, x))$, where $\overline{\mathbf{Q}}_\ell$, defined by $F \mapsto F_x$. Note here that $\overline{\mathbf{Q}}_\ell$ has the direct limit topology, where the limit is taken over finite extensions of $\mathbf{Q}_\ell$; because of this, each local system is actually one over a finite extension of $\overline{\mathbf{Q}}_\ell$.

Example 1.8 (Tate twist)**.** We ignored this for the most part in MATH731—Perverse Sheaves.

We define $\overline{\mathbf{Q}}_\ell(1) \coloneqq \overline{\mathbf{Q}}_\ell \otimes_{\mathbf{Z}_\ell} \mathbf{Z}_\ell(1)$, where

$$\mathbf{Z}_\ell(1) = \lim_n \mu_{\ell^n} = T_\ell(\overline{k}^*)$$

is the ℓ-adic Tate module.

Example 1.9. Let $k = \mathbf{F}_q$, and $\pi_1(\operatorname{Spec}(k)) = G_{\mathbf{F}_q} \cong \hat{\mathbf{Z}}$. Let $\operatorname{Frob}_{\mathbf{F}_1}$ be the geometric Frobenius (the *inverse* of raising functions to the qth power), which is an element of $G_{\mathbf{F}_q}$. Then, $\operatorname{Frob}_{\mathbf{F}_q} \circlearrowright \overline{\mathbf{Q}}_\ell(-1)$ by multiplication by q. The reason for this convention comes later when it will turn out to have positive weight.

1.2.3 Constructible sheaves

Local systems end up not being enough, so we expand our world to "constructible $\overline{\mathbf{Q}}_\ell$-sheaves": $\mathsf{Cons}(X) \coloneqq \mathsf{Cons}(X, \overline{\mathbf{Q}}_\ell)$. Note that $\mathsf{Cons}(X) \supset \mathsf{Loc}(X)$.

Facts 1.10.

1. If $F \in \mathsf{Cons}(X)$, there exists a stratification $\{Z_i \hookrightarrow X\}$ such that $F|_{Z_i} \in \mathsf{Loc}(Z_i)$.
2. *Dévissage:* if you want to prove things about constructible sheaves, you can reduce to the local system case. If $U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z$ open/closed decomposition, we get a short exact sequence
$$0 \longrightarrow j_!(F|_U) \longrightarrow F \longrightarrow i_*(F|_Z) \longrightarrow 0.$$
This will be useful to reduce the Weil conjectures to the case of local systems.
3. We have the following inclusions:
$$\begin{tikzcd}
\mathsf{Loc}(X) \arrow[r, phantom, "\subset"] & \mathsf{Cons}(X) \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\rotatebox{-90}{$\subseteq$}"] & \mathsf{Mod}(X, \overline{\mathbf{Q}}_\ell) \arrow[d, phantom, "\rotatebox{-90}{$\subseteq$}"] \\
& \mathsf{D}^b_{\mathsf{cons}}(X, \overline{\mathbf{Q}}_\ell) \arrow[r, phantom, "\subseteq"] \arrow[d, phantom, "\rotatebox{-90}{$=:$}"] & \mathsf{D}(\mathsf{Mod}(X, \overline{\mathbf{Q}}_\ell)) \\
& \mathsf{D}(X) &
\end{tikzcd}$$
If you want to see this stuff done precisely, you should look the pro-étale stuff. You can do this more categorically/intrinsically.
4. *The six functors.* Let X/k be a variety. Then,
   (a) There are bifunctors $- \otimes -$ and $\mathbf{R}\mathrm{Hom}(-,-)$ on $\mathsf{D}(X)$;
   (b) If $f\colon X \to Y$ is a morphism, there are functors
   $$f_!, f_*\colon \mathsf{D}(X) \to \mathsf{D}(Y), \qquad f^!, f^*\colon \mathsf{D}(Y) \to \mathsf{D}(X)$$
   such that $(f_!, f^!)$ and (f^*, f_*) form adjoint pairs (the left one in each ordered pair is a left adjoint, and the right one is a right adjoint).

   These work well with base change.

   Definition 1.11. Let $k = \overline{k}$, $K \in \mathsf{D}(X)$ (e.g., $K = \overline{\mathbf{Q}}_\ell$). Then,
   $$H^i(X, K) \coloneqq H^i(f_* K), \qquad H^i_c(X, K) \coloneqq H^i(f_! K),$$
   where f is the structure morphism $X \to \operatorname{Spec} k$.

   Note 1.12. If $X_0/\mathbf{F}_q$, $K \in \mathsf{D}(X_0)$, then both cohomology groups $H^i(X, K)$ and $H^i_c(X, K)$ get canonical actions of the absolute Galois group $G_{\mathbf{F}_q}$ by acting on the second factor K.

   Goal 1.13. We want to understand these representations of Galois groups on cohomology.
5. *Duality.* Let X/k be a variety, where k is a finite field or $k = \overline{k}$ (things might be bad when the field has infinite dimensional cohomology), and let $f\colon X \to \operatorname{Spec}(k)$. To do duality you need a dualizing object; in this case it will be the complex $\omega_X \coloneqq f^! \overline{\mathbf{Q}}_\ell \in \mathsf{D}(X)$, since the constant sheaf $\overline{\mathbf{Q}}_\ell$ is a dualizing object on $\operatorname{Spec}(k)$, and $f^!$ takes dualizing objects to dualizing objects. The complex ω_X satisfies:
   (a) $\mathcal{D}_X = \mathbf{R}\mathrm{Hom}(-, \omega_X)$ induces $\mathsf{D}(X)^{\mathsf{op}} \xrightarrow{\sim} \mathsf{D}(X)$.
   (b) If $f\colon X \to Y$ is a morphism, then
      - $\mathcal{D}_Y f_! = f_* \mathcal{D}_X$;
      - $\mathcal{D}_X f^! = f^* \mathcal{D}_Y$.
6. We record a computation: If X is smooth of dimension n, then $\omega_X = \overline{\mathbf{Q}}_\ell(n)[2n]$, so the dualizing object is the constant sheaf shifted by the (étale cohomological) dimension $2n$. This is similar to how in coherent cohomology for smooth (or even Gorenstein) varieties, the dualizing complex is shifted by the (coherent cohomological) dimension n.

Corollary 1.14 (Poincaré duality)**.** *If X is smooth of dimension n over $k = \overline{k}$, then*

$$H^i(X, \overline{\mathbf{Q}}_\ell) = H_c^{2n-i}(X, \overline{\mathbf{Q}}_\ell(n))^\vee = \left(H_c^{2n-i}(X, \overline{\mathbf{Q}}_\ell)\right)^\vee(-n).$$

This looks like the regular Poincaré duality except it has a Tate twist; this also shows up when you do duality with Hodge structures.

Proof of Corollary. Set $K = \overline{\mathbf{Q}}_\ell$, and $f\colon X \to \operatorname{Spec}(k)$. Then,

$$\mathcal{D}_{\mathrm{pt}}(f_* K) = f_! \mathcal{D}_X(K) = f_! \, \mathbf{R}\mathrm{Hom}(K, \mathbf{Q}_\ell(n)[2n]) = f_! \mathbf{Q}_\ell(n)[2n]$$

Taking H^{-i} on both sides, and using the definition $H^i(X, K) = H^i(f_* K)$ and similarly for compactly supported cohomology, we get the statement desired. $\square$

We should have mentioned at some point that $\mathsf{Cons}(\mathrm{pt})$ is the same as finite-dimensional vector spaces.

Example 1.15. Let $X_0/\mathbf{F}_q$ be a smooth affine curve, and $F \in \mathsf{Loc}(X_0)$. We will later see that this is the key computation: you can reduce to the case of curves, but with interesting coefficients.

Theorem 1.16 (Artin)**.** *$H^i(X, F) = 0$ for all $i > 1$, i.e., the only interesting groups are $H^0(X, F)$ and $H^1(X, F)$.*

One of these is easy by using that $\pi_1^{\mathrm{geom}}(X) \subset \pi_1^{\mathrm{arith}}(X)$, and the statement about representations of fundamental groups we had before:

$$H^0(X, F) = F^{\pi_1^{\mathrm{geom}}(X)}.$$

$H^1(X, F)$ is a bit more interesting…
The dual statement for compactly supported cohomology is:

$$H_c^i(X, F) = \begin{cases} 0 & i \le 0,\ i \ge 3 \\ F_{\pi_1^{\mathrm{geom}}(X)} & i = 2 \\ ?? & i = 1 \end{cases}$$

7. *Lefschetz trace formula.* This is not how it is discussed in SGA, but this is a good way to think of it nowadays. Let $X_0/\mathbf{F}_q$ be a variety, and consider the "sheaf-function correspondence":

$$\begin{aligned} \mathsf{D}(X_0) &\xrightarrow{\phi} \mathsf{Fun}(X_0(\mathbf{F}_q), \overline{\mathbf{Q}}_\ell) \\ K &\longmapsto \big((x\colon \operatorname{Spec}(\mathbf{F}_q) \to X_0) \mapsto \operatorname{Trace}(\mathrm{Frob}_x \mid x^* K)\big) \end{aligned}$$

where if $K \in \mathsf{D}(\overline{\mathbf{Q}}_\ell\text{-vector spaces})$, and $g\colon K \to K$, then

$$\operatorname{Trace}(g \mid K) = \sum (-1)^i \operatorname{Trace}(g \mid H^i(K)),$$

where the signs are there to make Trace work well with short exact sequences.
The Lefschetz trace formula says this definition works well with pushforward and pullback.

Theorem 1.17 (Lefschetz trace formula)**.** *Let $f\colon X_0 \to Y_0$ be a morphism over $\mathbf{F}_q$. Then,*
(a) ϕ commutes with f^* and extension of scalars to $\mathbf{F}_{q^r}$;
(b) ϕ commutes with $f_!$:

$$\begin{tikzcd} \mathsf{D}(X_0) \arrow[r, "\phi"] \arrow[d, "f_!"] & \mathsf{Fun}(X_0(\mathbf{F}_q), \overline{\mathbf{Q}}_\ell) \arrow[d, "\text{sum } \phi \text{ along fibres}"] \\ \mathsf{D}(Y_0) \arrow[r, "\phi"] & \mathsf{Fun}(Y_0(\mathbf{F}_q), \overline{\mathbf{Q}}_\ell) \end{tikzcd}$$

where $\mathsf{Fun}(-,-)$ denotes functions as sets.

Example 1.18. $Y_0 = \mathrm{Spec}(\mathbf{F}_q)$, $K = \overline{\mathbf{Q}}_\ell \in \mathsf{D}(X_0)$, and the theorem says

$$\begin{tikzcd}
\mathsf{D}(X_0) \arrow[r, "\phi"] \arrow[d, "f_!"] & \mathsf{Fun}(X_0(\mathbf{F}_q), \overline{\mathbf{Q}}_\ell) \arrow[d, "f_!"] \\
\mathsf{D}(Y_0) \arrow[r, "\phi"] & \overline{\mathbf{Q}}_\ell
\end{tikzcd}$$

commutes, and so

$$\begin{tikzcd}
f_!\phi(K) \arrow[r, equal] \arrow[d, equal] & \phi(f_!(K)) \arrow[d, equal] \\
\displaystyle\sum_{x \in X_0(\mathbf{F}_q)} 1 \arrow[d, equal] & \mathrm{Trace}(\mathrm{Frob}_{\mathbf{F}_q} \mid \mathbf{R}\Gamma_c(X, \overline{\mathbf{Q}}_\ell) \arrow[d, equal] \\
\#X_0(\mathbf{F}_q) & \sum (-1)^i \,\mathrm{Trace}(\mathrm{Frob}_{\mathbf{F}_q} \mid H^i_c(X, \overline{\mathbf{Q}}_\ell)
\end{tikzcd}$$

In this way, the *one* complex $H^i_c(X, \overline{\mathbf{Q}}_\ell)$ contains the information about all $\mathbf{F}_{q^r}$-points:

$$Z(X_0, t) = \prod_{i=0}^{2\dim(X)} \Big(\det\big(1 - t \cdot \mathrm{Frob}_{\mathbf{F}_q} \bigm| H^i_c(X, \overline{\mathbf{Q}}_\ell)\big)\Big)^{(-1)^{i+1}}. \tag{1}$$

Exercise 1.19. Prove rationality of $Z(X_0, t)$ using this formula.

Similarly, the functional equation comes from Poincaré duality, and the Betti numbers come from smooth and proper base change.

All this stuff came *before* the papers we will talk about in this seminar.

1.3 Deligne's theorem

The theorem discusses the eigenvalues of the operator we have on the right hand side in (1).

Fix $\iota\colon \overline{\mathbf{Q}}_\ell \hookrightarrow \mathbf{C}$. We can talk about $|\iota(\alpha)| = |\alpha|_\iota$.

Definition 1.20.

1. Let $V \in \mathsf{Rep}^{\mathsf{cont}}_{\overline{\mathbf{Q}}_\ell}(G_{\mathbf{F}_q})$. Then, V is *ι-pure of weight w* if

$$\left|\begin{matrix}\text{each eigenvalue} \\ \text{of } \mathrm{Frob}_{\mathbf{F}_q} \circlearrowright V\end{matrix}\right|_\iota = \sqrt{(\#\mathbf{F}_q)^w} = q^{w/2} \tag{$*$}$$

2. If $X_0/\mathbf{F}_q$ is smooth and geometrically connected, and $F \in \mathsf{Loc}(X)$, then F is *ι-pure of weight w* if for all $x\colon \mathrm{Spec}(\mathbf{F}_{q^r}) \to X_0$, the pullback x^*F is ι-pure of weight w.
3. In the setup of 1 or 2, F is *ι-mixed of weight $\leq w$* (or respectively $\geq w$), if have $\leq$ (respectively $\geq w$) in $(*)$.

Example 1.21. Let X_0 be a smooth, geometrically connected variety of dimension n. Let $F \in \mathsf{Loc}(X)$ be ι-pure of weight w. Then, $H^0(X, F) \supset H^0_c(X, F)$ are both ι-pure of weight w.

Proof. Choose $x\colon \mathrm{Spec}(\mathbf{F}_{q^r}) \to X_0$, and use $H^0(X, F) \subset F_x$. F_x has the statement for all eigenvalues, so $H^0(X, F)$ does as well. □

Duality gives that $H^{2n}_c(X, F) = H^0(X, F^\vee)^\vee(-n)$ is ι-pure of weight $w + 2n$.

This is the babiest case of Deligne's theorem; the theorem also gives information for intermediate cohomology groups.

Theorem 1.22 (Deligne)**.** *Let $X_0/\mathbf{F}_q$ be a variety (not necessarily projective), and let $F \in \mathsf{Loc}(X_0)$ be ι-pure of weight w. Then, $H^i_c(X, F)$ is ι-mixed of weight $\leq w + i$.*

A small (purely geometric) argument (to be discussed later) shows that this is equivalent to the same statement for X_0 a smooth geometrically connected affine curve (you can even use $\mathbf{A}^1$!), and $i = 1$.

Corollary 1.23. *If X_0 is a smooth, projective, geometrically connected variety over $\mathbf{F}_q$, then $H^i(X, \overline{\mathbf{Q}}_\ell)$ is ι-pure of weight i.*

Proof of Corollary. First, $H^i(X, \overline{\mathbf{Q}}_\ell) = H^i_c(X, \overline{\mathbf{Q}}_\ell)$ since X is complete, which is ι-mixed of weight $\leq i$ by the Theorem. Second, by duality, $H^i(X, \overline{\mathbf{Q}}_\ell) = H^{2n-i}(X, \overline{\mathbf{Q}}_\ell^\vee)^\vee(-n)$, and the right hand side is ι-mixed of weight $\geq i$ by the Theorem again. □

2 May 11—Weil Sheaves (Tyler Foster)

The first thing we need to do is to talk about different actions of Frobenius; it will be important to keep track of these things when we define Weil sheaves and state the Grothendieck trace formulas.

Fix a ground field $\mathbf{F}_q = \kappa$, and fix an algebraic closure $k = \mathbf{F}_q^{\text{alg}}$. The *Frobenius morphism* $\sigma_k \colon k \to k$ is defined by $a \mapsto a^q$. The *geometric Frobenius* is $F = \sigma^{-1}$. The *Galois group* is $\mathrm{Gal}(k/\mathbf{F}_q) \cong \hat{\mathbf{Z}}$, which contains σ as a topological generator. The *Weil group* $W(k/\mathbf{F}_q) = \langle F \rangle \cong \mathbf{Z}$ is the subgroup of $\mathrm{Gal}(k/\mathbf{F}_q)$ generated by the geometric Frobenius.

We will consider different Frobenius operators on schemes. The first one is the familiar one: Given a $\mathbf{F}_q$-scheme X, the map σ induces a morphism $\sigma_{X/\mathbf{F}_q} \colon X \to X$ which fixes the underlying topological space $|X|$ of X, and acts by qth powers $f \mapsto f^q$ on the structure sheaf $\mathcal{O}_X$.

Now consider an $\mathbf{F}_q$-scheme X_0. We get a second Frobenius action

$$\begin{tikzcd}
X \arrow[dr, dashed, "\mathrm{Fr}_X", "\exists!"'] \arrow[drr, bend left, "\sigma_{X/\mathbf{F}_q}"] \arrow[dd, bend right, "\pi"'] & & & \\
& X \arrow[r] \arrow[d] \arrow[dr, dashed, "\sigma_{k/\mathbf{F}_q} \times \mathrm{id}_{X_0}"] & X_0 \arrow[dr, equal] & \\
& \mathrm{Spec}\, k \arrow[dr, "\sigma_{k/\mathbf{F}_q}"'] & X \arrow[r] \arrow[d, "\pi"] \arrow[dr, phantom, "\lrcorner"] & X_0 \arrow[d, "\pi_0"] \\
& & \mathrm{Spec}\, k \arrow[r] & \mathrm{Spec}\, \mathbf{F}_q
\end{tikzcd} \tag{2}$$

We call this morphism $\mathrm{Fr}_X \colon X \to X$, the *Frobenius endomorphism of X*. Note it is the unique map such that $\sigma_{k/\mathbf{F}_q} \times \mathrm{id}_{X_0} \circ \mathrm{Fr}_X = \sigma_{X/\mathbf{F}_q}$ by using the larger cartesian parallelogram.

Example 2.1. Let $X_0 = \mathbf{A}^1_{\mathbf{F}_q}$. We are interested in the factorization

$$\begin{tikzcd}
\mathbf{A}^1_k \arrow[r, "\mathrm{Fr}_{\mathbf{A}^1_k}"'] \arrow[rr, bend left, "\sigma_{\mathbf{A}^1_k/\mathbf{F}_q}"] & \mathbf{A}^1_k \arrow[r, "\sigma_k \times \mathrm{id}"'] & \mathbf{A}^1_k
\end{tikzcd}$$

which is dual to

$$\begin{tikzcd}
k[t] & k[t] \arrow[l] & k[t] \arrow[l] \arrow[ll, bend right, "f^q \mapsfrom f"'] \\
t^q & t \arrow[l, mapsto] & a^q \quad \mathrm{coeff}\ a \arrow[l, phantom]
\end{tikzcd}$$

$$t^q \longleftarrow\!\shortmid\; t \qquad a^q \longleftarrow\!\shortmid\; \text{coeff } a$$

Now draw the same bottom right square as in (2), but with F replacing $\sigma_{k/\mathbf{F}_q}$:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, dashed, "F\times \mathrm{id}_{X_0}"] & X_0 \arrow[dr, equal] & \\
\operatorname{Spec} k \arrow[dr, "F"'] & X \arrow[r] \arrow[d, "\pi"] \arrow[dr, phantom, "\lrcorner" very near start] & X_0 \arrow[d, "\pi_0"] \\
& \operatorname{Spec} k \arrow[r] & \operatorname{Spec} \mathbf{F}_q
\end{tikzcd}$$

This gives the *Frobenius automorphism of X* $F_X \colon X \to X$ where $\mathrm{Fr}_X = F_x \cdot \mathrm{id}_{X_0}$.

We will discuss Grothendieck's trace formula in terms of these Frobenius morphisms; Weil sheaves will provide a language to extend it.

Fix X_0 a scheme over $\mathbf{F}_q$, and fix $\mathscr{G}_0$ an (étale) $\overline{\mathbf{Q}}_\ell$-sheaf. Also fix $x \in |X_0|$ a closed point with residue field $k(x)$ and set $d(x) = [k(x) : \mathbf{F}_q]$, and a geometric point

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[r, "\overline{x}"] \arrow[d] \arrow[dr, "\overline{x}"] & X \arrow[d, "\pi"] \\
\operatorname{Spec} k(x) \arrow[r, "x"] & X_0
\end{tikzcd}$$

Now form the fibre $\mathscr{G}_{0\overline{x}}$ which comes with an action of geometric Frobenius $F \colon \mathscr{G}_{0\overline{x}} \to \mathscr{G}_{0F(\overline{x})}$, with $d(x)$th power $F_x \colon \mathscr{G}_{0\overline{x}} \to \mathscr{G}_{0\overline{x}}$.

Now recall that an (étale) $\overline{\mathbf{Q}}_\ell$-sheaf is represented by an étale E-sheaf, where E is some finite field extension of $\mathbf{Q}_\ell$:

$$\mathbf{Q}_\ell \underset{\text{fin}}{\subset} E \subset \overline{Q}_\ell$$

and we have an inverse system of finite étale E-sheaves $(\mathscr{G}_{0i})_{i=1}^{\infty}$, which represent $\mathscr{G}_0$ as a cokernel

$$\begin{tikzcd}
R_i \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr, "\text{ét}"'] & & G_{0i} \arrow[dl, "\text{ét}"] \\
& X_0 &
\end{tikzcd}$$

We then have the following commutative diagram

$$\begin{tikzcd}
G_{0i} \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "F_{G_{0i}}"] \arrow[ddr, bend right] & & \\
& F_X^* G_{0i} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & G_{0i} \arrow[d] \\
& X \arrow[r, "F_X"] & X
\end{tikzcd}$$

giving rise to a morphism $\mathscr{G} \to F_X^* \mathscr{G}$.

We now claim that there is an isomorphism $F_X^* \mathscr{G} \xrightarrow{\sim} \mathscr{G}$. First, we have two projection morphisms

$$\begin{tikzcd}
X \times_{X_0} X \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_2"'] & X \arrow[r] & X_0
\end{tikzcd}$$

whose compositions to X_0 are equal, giving an isomorphism $\mathrm{pr}_1^* \mathscr{G} \xrightarrow{\sim} \mathrm{pr}_2^* \mathscr{G}$. Now note that

$$X \times_{X_0} X \cong (k \otimes_{\mathbf{F}_q} k) \otimes_{\mathbf{F}_q} X_0 \cong \mathrm{Gal}(k, \mathbf{F}_q) \times X =: \coprod_{\mathrm{Gal}(k/\mathbf{F}_q)} X.$$

Replacing $X \times_{X_0} X$ with $\mathrm{Gal}(k, \mathbf{F}_q) \times X$ in the diagram above, we get the new diagram

$$\begin{tikzcd}
\mathrm{Gal}(k, \mathbf{F}_q) \times X \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\mathrm{pr}_2"'] & X \arrow[r] & X_0
\end{tikzcd}$$

where α denotes the action of $\mathrm{Gal}(k/\mathbf{F}_q)$ via F_X. The isomorphism $\mathrm{pr}_1^* \mathscr{G} \xrightarrow{\sim} \mathrm{pr}_2^* \mathscr{G}$ from above breaks up into $\# \mathrm{Gal}(k/\mathbf{F}_q)$ copies of an isomorphism $g^*\mathscr{G} \xrightarrow{\sim} \mathscr{G}$ for each $g \in \mathrm{Gal}(k/\mathbf{F}_q)$. In particular, for $g \in \mathrm{Gal}(k/\mathbf{F}_q)$ corresponding to the Frobenius morphism F_X, we have an isomorphism $F_X^*\mathscr{G} \xrightarrow{\sim} \mathscr{G}$.

By precomposition with the morphism $\mathscr{G} \to F_X^*\mathscr{G}$ from before, we obtain an endomorphism $\mathscr{G} \to F_X^*\mathscr{G} \xrightarrow{\sim} \mathscr{G}$. We will look at this morphism fibrewise.

Now we define the L-function for $\mathscr{G}_0$ to be

$$L(X_0, \mathscr{G}_0, t) = \prod_{x \in |X_0|} \det\Big(1 - t^{d(x)} F_x,\ \mathscr{G}_{0\overline{x}}\Big)^{-1}$$

By using the Frobenius endomorphism $\mathrm{Fr}_X \colon X \to X$, and by unpacking what a $\overline{\mathbf{Q}}_\ell$-sheaf is as before, we have an isomorphism $\alpha \colon \mathscr{G} \xrightarrow{\sim} \mathrm{Fr}_X^* \mathscr{G}$ with inverse $\alpha^{-1} \colon \mathrm{Fr}_X^* \mathscr{G} \xrightarrow{\sim} \mathscr{G}$. Because $\mathrm{Fr}_X \colon X \to X$ is proper, we get an induced map on compactly supported cohomology:

$$\begin{tikzcd}
H_c^i(X, \mathscr{G}) \arrow[r, "\mathrm{Fr}_X^*"'] \arrow[rr, bend left, "F"] & H_c^i(X, \mathrm{Fr}_X^* \mathscr{G}) \arrow[r, "\alpha^{-1}"'] & H_c^i(X, \mathscr{G})
\end{tikzcd}$$

Theorem 2.2 (Grothendieck trace formula)**.** *The L-function can be written as a finite product:*

$$L(X_0, \mathscr{G}_0, t) = \prod_{i=0}^{2\dim X} \det\big(1 - tF,\ H_c^i(X, \mathscr{G})\big)^{(-1)^{(i+1)}}.$$

The key piece of structure that lets us formulate this trace formula, and even just define the L-function itself, is the action of Frobenius on stalks of our sheaf, which exists since we started with a $\overline{\mathbf{Q}}_\ell$-sheaf that lives on X_0. The definition of a Weil sheaf just introduces this action of Frobenius directly, so that it still makes sense to write down its L-function.

Definition 2.3. A *Weil sheaf* on X_0 consists of

1. A $\overline{\mathbf{Q}}_\ell$-sheaf $\mathscr{G}$ on X;
2. An isomorphism $F^* \colon F_X^*\mathscr{G} \xrightarrow{\sim} \mathscr{G}$.

Notation 2.4. We will refer to a Weil sheaf $\mathscr{G}_0$ as "$\mathscr{G}_0$ on X_0", even though the actual sheaf lives on X.

Definition 2.5. A Weil sheaf $\mathscr{G}_0$ on X_0 is *smooth of rank r* if $\mathscr{G}$ is smooth of rank r on X. Bhargav called these *lisse* in MATH731—Perverse Sheaves.

Properties 2.6. Here are some properties of Weil sheaves:

1. Weil sheaves form an abelian category with étale $\overline{\mathbf{Q}}_\ell$-sheaves on X_0 as a full subcategory;
2. The category does not depend on $\mathbf{F}_q$, that is, given $\kappa' \subset \mathbf{F}_q$, restriction of scalars gives a natural equivalence between Weil sheaves on $X_0/\mathbf{F}_q$ to Weil sheaves on X_0/κ';
3. Weil sheaves have pullbacks, derived direct images, and direct image with compact support;
4. F^* induces a morphism $F \colon H_c^i(X, \mathscr{G}) \to H_c^i(X, \mathscr{G})$;
5. Any Weil sheaf comes with an automorphism $F_x \colon \mathscr{G}_{\overline{x}} \xrightarrow{\sim} \mathscr{G}_{\overline{x}}$ for a geometric point $\overline{x}$ over a closed point $x \in |X_0|$.

Remark 2.7. In future sections, "sheaf" means Weil sheaf, particularly in [KW01, Chap. I].

We now want to show that the Grothendieck trace formula also holds for these Weil sheaves. To do so, we need to discuss a Tannakian duality between Weil sheaves and representations of $\pi_1(X_0, \overline{a})$.

Assume X_0 is geometrically connected, and fix a geometric point

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[r, "\overline{a}"] \arrow[dr, "\overline{a}"'] & X_0 \\
& X \arrow[u]
\end{tikzcd}$$

We then have the *monodromy exact sequence*

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X,\overline{a}) \arrow[r] & \pi_1(X_0,\overline{a}) \arrow[r] & \mathrm{Gal}(k/\mathbf{F}_q) \arrow[r] & 1 \\
1 \arrow[r] & \pi_1(X,\overline{a}) \arrow[r] \arrow[u, equal] & W(X_0,\overline{a}) \arrow[r] \arrow[u, hook] \arrow[dr, "\deg"'] & W(k/\mathbf{F}_q) \arrow[r] \arrow[u, phantom, "\cup"] & 1 \\
& & & \mathbf{Z} \arrow[u, phantom, "\wr\|"] &
\end{tikzcd}$$

Note that $W(k/\mathbf{F}_q) \cong \mathbf{Z}$ is *not* given the subspace topology relative to $\mathrm{Gal}(k/\mathbf{F}_q) \cong \hat{\mathbf{Z}}$. The group $W(X_0,\overline{a})$ is called the *Weil group of X_0 attached to the base point $\overline{a}$.*

To understand the difference between $\overline{\mathbf{Q}}_\ell$-sheaves and Weil sheaves, we state the Tannakian duality in both contexts. For $\overline{\mathbf{Q}}_\ell$-sheaves, we have

$$\left\{\text{étale } \overline{\mathbf{Q}}_\ell\text{-sheaves}\right\} \xrightarrow{\mathrm{fib}_{\overline{a}}} \left\{\begin{matrix}\text{continuous representations of} \\ \pi_1(X_0,\overline{a}) \text{ on } \overline{\mathbf{Q}}_\ell\text{-vector spaces}\end{matrix}\right\}$$

where *continuous* means that there exists a finite extension $\mathbf{Q}_\ell \subset E \subset \overline{\mathbf{Q}}_\ell$ with E-linear subspace $W \subset V$ such that $V = \overline{\mathbf{Q}}_\ell \otimes_E W$ and $\pi_1(X_0,\overline{a})$ acts continuously on W (this is the same as saying the action is continuous by the topology of $\overline{\mathbf{Q}}_\ell$). This induces an equivalence

$$\left\{\text{smooth étale } \overline{\mathbf{Q}}_\ell\text{-sheaves}\right\} \underset{\sim}{\xrightarrow{\mathrm{fib}_{\overline{a}}}} \left\{\begin{matrix}\text{finite dimensional continuous} \\ \text{representations of } \pi_1(X,\overline{a}) \\ \text{on } \overline{\mathbf{Q}}_\ell\text{-vector spaces}\end{matrix}\right\}$$

For Weil sheaves, we instead have

$$\left\{\text{Weil sheaves}\right\} \xrightarrow{\mathrm{fib}_{\overline{a}}} \left\{\begin{matrix}\text{continuous representations of} \\ W(X_0,\overline{a}) \text{ on } \overline{\mathbf{Q}}_\ell\text{-vector spaces}\end{matrix}\right\}$$

and restricting to smooth objects, we have

$$\left\{\text{smooth Weil sheaves}\right\} \underset{\sim}{\xrightarrow{\mathrm{fib}_{\overline{a}}}} \left\{\begin{matrix}\text{finite dimensional continuous} \\ \text{representations of } W(X,\overline{a}) \\ \text{on } \overline{\mathbf{Q}}_\ell\text{-vector spaces}\end{matrix}\right\}$$

Special Case 2.8. Smooth rank 1 Weil sheaves on $\operatorname{Spec} \mathbf{F}_q$ are the same thing as characters

$$\begin{tikzcd}
\phi\colon W(k/\mathbf{F}_q) \arrow[r] \arrow[d, phantom, "\wr\|"] & \overline{\mathbf{Q}}_\ell^* \\
\mathbf{Z} & \\
F \arrow[r, mapsto] & \phi(F) = \mathfrak{b}
\end{tikzcd}$$

and conversely, any $\mathfrak{b} \in \overline{\mathbf{Q}}_\ell^*$ gives a Weil sheaf $\mathscr{L}_\mathfrak{b}$ on $\operatorname{Spec} \mathbf{F}_q$. We will also use $\mathscr{L}_\mathfrak{b}$ to denote the pullback of this sheaf to X_0.

We have another criterion to determine if a Weil sheaf is not a $\overline{\mathbf{Q}}_\ell$-sheaf, in addition to the duality statement from before:

Theorem 2.9. *Let X_0 be a scheme over $\mathbf{F}_q$, and let $\mathscr{G}_0 = (F_X^*\mathscr{G} \xrightarrow{\sim} \mathscr{G})$ be a Weil sheaf on X_0. Then,*

1. *If X_0 is normal and geometrically connected, and if $\mathscr{G}_0$ is irreducible and smooth of rank r, then $\mathscr{G}_0$ is an étale $\overline{\mathbf{Q}}_\ell$-sheaf on X_0 if and only if $\bigwedge^r \mathscr{G}_0$ is an étale sheaf.*

   Corollary 2.10. *For any smooth, irreducible sheaf $\mathscr{G}_0$, there exists some $\mathscr{L}_{\mathfrak{b}}$ and some $\mathscr{F}_0$ an étale sheaf such that $\mathscr{G}_0 \cong \mathscr{F}_0 \otimes \mathscr{L}_{\mathfrak{b}}$.*

2. *For a general smooth Weil sheaf $\mathscr{G}_0$ on a normal, geometrically connected X_0, there exists a filtration*

   $$0 = \mathscr{G}_0^{(0)} \subset \mathscr{G}_0^{(1)} \subset \cdots \subset \mathscr{G}_0^{(r)} = \mathscr{G}_0$$

   where $\mathscr{G}_0^{(j)}/\mathscr{G}_0^{(j-1)} \cong \mathscr{F}_0^{(j)} \otimes \mathscr{L}_{\mathfrak{b}_j}$ where $\mathscr{F}_0^{(j)}$ is smooth étale, and $\mathscr{L}_{\mathfrak{b}_j}$ is a Weil sheaf.

   Corollary 2.11 (Grothendieck trace formula for Weil sheaves)**.** *Given a smooth Weil sheaf $\mathscr{G}_0$ on X_0, define*

   $$L(X_0, \mathscr{G}_0, t) = \prod_{x \in |X_0|} \det\Bigl(1 - t^{d(x)} F_x,\ \mathscr{G}_{\overline{x}}\Bigr)^{-1}. \tag{3}$$

   Then, we can compute the L-function as

   $$L(X_0, \mathscr{G}_0, t) = \prod_{i=0}^{2\dim X} \det\bigl(1 - tF,\ H_c^i(X, \mathscr{G})\bigr)^{(-1)^{(i+1)}} \tag{4}$$

   Proof of Corollary. In the irreducible case, $\mathscr{G}_0 = \mathscr{G}_0 \otimes \mathscr{L}_{\mathfrak{b}}$ and $\mathscr{G}_{\overline{x}} = \mathscr{F}_{0\overline{x}} \otimes \mathscr{L}_{\mathfrak{b}\overline{x}}$. We can rewrite each factor on the right-hand side of (3) as

   $$\det\Bigl(1 - t^{d(x)} \mathfrak{b}^{d(x)} F_x,\ \mathscr{F}_{0\overline{x}}\Bigr).$$

   For (4), each factor becomes

   $$\det\bigl(1 - tF,\ H_c^i(X, \mathscr{F} \otimes \mathscr{L}_{\mathfrak{b}})\bigr) = \det\bigl(1 - t\mathfrak{b}F,\ H_c^i(X, \mathscr{F})\bigr)$$

   since $\mathscr{L}_{\mathfrak{b}}$ is the pullback of a Weil sheaf on $\operatorname{Spec} \mathbf{F}_q$. Grothendieck trace formula for $L(X_0, \mathscr{F}_0, \mathfrak{b}t)$ gives the irreducible case.
   For the general case, use the filtration in (2) and the multiplicativity of trace on filtrations. ☐

3 May 18—Weights I (Brandon Carter)

Notation 3.1. Unless otherwise stated, $\kappa = \mathbf{F}_q$, $k = \overline{\kappa}$, X_0 is an algebraic (that is, finite type) scheme over κ, $\mathscr{G}_0$ is a (Weil) sheaf on X_0, and $\tau\colon \overline{\mathbf{Q}}_\ell \xrightarrow{\sim} \mathbf{C}$ is a chosen isomorphism. For a (closed) point $x \in |X_0|$, we also denote $d(x) = [\kappa(x) : \kappa]$, and $N(x) = \#\kappa(x) = q^{d(x)}$.

Definition 3.2. Let $\beta \in \mathbf{R}$. Then,

1. For each closed point $x \in |X_0|$, fix $\overline{x}$ a geometric point lying over x. Then, we have the Weil group $W(k/\kappa(x))$ acting on the stalk $\mathscr{G}_{0\overline{x}}$. We say that $\mathscr{G}_0$ is *τ-pure* of weight β if for all $x \in |X_0|$, we have that for all eigenvalues α of the geometric Frobenius morphism $F_x\colon \mathscr{G}_{0\overline{x}} \to \mathscr{G}_{0\overline{x}}$, we have $|\tau(\alpha)|^2 = N(x)^\beta$.
2. We say $\mathscr{G}_0$ is *τ-mixed* if there is a finite filtration of subsheaves

   $$0 = \mathscr{G}_0^{(0)} \subset \mathscr{G}_0^{(1)} \subset \cdots \subset \mathscr{G}_0^{(r)} = \mathscr{G}_0$$

   such that $\mathscr{G}_0^{(j)}/\mathscr{G}_0^{(j-1)}$ is τ-pure of some weight.
3. $\mathscr{G}_0$ is (pointwise) *pure* of weight β if is τ-pure of weight β for all choices $\tau\colon \overline{\mathbf{Q}}_\ell \xrightarrow{\sim} \mathbf{C}$.
4. $\mathscr{G}_0$ is *mixed* if there exists a finite filtration as in (2) with successive quotients being pure.

Note some higher rank vector bundles are neither τ-pure nor τ-mixed.

Remark 3.3. If $\mathscr{G}_0$ is τ-pure for every τ, but the weight depends on τ, then there exists some $\mathfrak{b} \in \overline{\mathbf{Q}}_\ell$ such that $\mathscr{G}_0 \cong \mathscr{F}_0 \otimes \mathscr{L}_\mathfrak{b}$ with $\mathscr{F}_0$ being pure.

Permanence Properties 3.4.

1. If $f_0 \colon X_0 \to Y_0$ is a morphism over κ, and $\mathscr{G}_0$ is a sheaf on Y_0, then
    - $f_0^*(\mathscr{G}_0)$ is τ-pure of weight β if $\mathscr{G}_0$ is τ-pure of weight β.
    - If f is surjective, we get an if and only if statement.
2. If $f_0 \colon X_0 \to Y_0$ is finite, and $\mathscr{G}_0$ is a sheaf on X_0, then $f_{0*}(\mathscr{G}_0)$ is τ-pure of weight β if $\mathscr{G}_0$ is.
3. If X_0/κ, $\mathscr{G}_0$ a sheaf on X_0, and κ'/κ a finite extension, then $\mathscr{G}_0$ is pure of weight β if and only if the pullback on $X_0 \otimes_\kappa \kappa'$ is.

Note that (1) and (2) imply (3).

Remark 3.5. Similar statements for pure and τ-mixed sheaves hold, except for one exception: for τ-mixed sheaves, the forward direction of the second subbullet of (1) only holds for *finite* maps in general. [The filtration is probably not preserved.]

Definition 3.6. For X_0 and $\mathscr{G}_0$ as before, and for fixed τ, we define

$$w(\mathscr{G}_0) = \sup_{x \in |X_0|} \sup_{\substack{\alpha \\ \text{eigenvalue}}} \frac{\log\bigl(|\tau(\alpha)|^2\bigr)}{\log(N(x))}.$$

If $\mathscr{G}_0$ is trivial (i.e., the zero sheaf), set $w(\mathscr{G}_0) = -\infty$.

We can use weights to talk about zeros and poles of the L-function.

Lemma 3.7. *Suppose $\mathscr{G}_0$ is a sheaf on X_0, such that $w(\mathscr{G}_0) \le \beta$, i.e., for all $x \in |X_0|$, and α an eigenvalue of $F_x \colon \mathscr{G}_{0\overline{x}} \to \mathscr{G}_{0\overline{x}}$, we have*

$$|\tau(\alpha)|^2 \le N(x)^\beta = q^{d(x)\beta}.$$

Then, the L-function

$$\tau L(X_0, \mathscr{G}_0, t) = \prod_{x \in |X_0|} \tau \det\Bigl(1 - t^{d(x)} F_x,\ \mathscr{G}_{0\overline{x}}\Bigr)^{-1}$$

converges for all $|t| < q^{-\beta/2 - \dim(X_0)}$ and has no zeros or poles in this region.

The idea is that since Grothendieck's trace formula says that the L-function is already a meromorphic function, we can use the logarithmic derivative to detect zeroes and poles of the L-function.

Proof. Reduce to the case where X_0 is affine, reduced, and irreducible. Then,

$$\frac{\tau L'(X_0, \mathscr{G}_0, t)}{L(X_0, \mathscr{G}_0, t)} = \sum_{x \in |X_0|} \sum_{n=1}^{\infty} d(x) \operatorname{Tr}(F_x^n) t^{d(x)n-1} = \sum_{n=1}^{\infty} \Biggl(\sum_{\substack{x \in |X_0| \\ d(x) \mid n}} d(x) \operatorname{Tr}\Bigl(F_x^{n/d(x)}\Bigr) \Biggr) t^{n-1},$$

where the second equality is by changing the order of summation, and also changing the index of summation n to $n/d(x)$. Now by assumption on $w(\mathscr{G}_0)$, we have a trivial bound on the traces that appear in this sum:

$$\Bigl| \tau \operatorname{Tr}\Bigl(F_x^{n/d(x)}\Bigr) \Bigr| \le r \cdot q^{n\beta/2} \quad \text{where} \quad r = \max_{x \in |X_0|} \dim_{\overline{\mathbf{Q}}_\ell} \mathscr{G}_{0\overline{x}}.$$

This gives the bound

$$\frac{\tau L'(X_0, \mathscr{G}_0, t)}{L(X_0, \mathscr{G}_0, t)} \le \sum_{n=1}^{\infty} \Biggl(\sum_{\substack{x \in |X_0| \\ d(x) \mid n}} d(x) \Biggr) r \cdot q^{n\beta/2} t^{n-1} = \sum_{n=1}^{\infty} \#X_0(\mathbf{F}_{q^n}) \cdot r q^{n\beta/2} t^{n-1}.$$

Now Noether normalization implies $\#X_0(\mathbf{F}_{q^n}) \leq C \cdot q^{n \dim X_0}$ for some constant C, so

$$\frac{\tau L'(X_0, \mathscr{G}_0, t)}{L(X_0, \mathscr{G}_0, t)} \leq \sum_{n=1}^{\infty} C \cdot r q^{n(\dim X_0 + \beta/2)} t^{n-1},$$

which is just a geometric series. ☐

Lemma 3.8. *Let X_0 be a smooth irreducible curve over κ, with $U_0 \overset{j_0}{\hookrightarrow} X_0$ a nonempty open subset. Denote $S_0 = X_0 \setminus U_0$ to be the complement of U_0. Let $\mathscr{G}_0$ be a sheaf on X_0 such that the restriction $j_0^*\mathscr{G}_0$ is smooth and $H_S^0(X, \mathscr{G}) = 0$, i.e., $\mathscr{G}$ has no sections supported on the complement $S = X \setminus U$. Then, $w(j_0^*(\mathscr{G}_0)) \leq \beta$ implies $w(\mathscr{G}_0) \leq \beta$.*

Sketch of Proof. We denote $\mathscr{F}_0 = j_0^*(\mathscr{G}_0)$ in the following.

1. Reduce to the case where X_0 is affine and geometrically irreducible, and $j_{0*}j_0^*\mathscr{G}_0 = \mathscr{G}_0$. Then, $H_c^0(X, \mathscr{G}) = 0$ (by using $j_{0*}j_0^*\mathscr{G}_0 = \mathscr{G}_0$ and affinity), and so we have

$$L(X_0, \mathscr{G}_0, t) = L(U_0, j_0^*(\mathscr{G}_0), t) \cdot \prod_{s \in |S_0|} \det\Big(1 - t^{d(s)} F_s,\ \mathscr{G}_{0\overline{s}}\Big)^{-1} = \frac{\det\big(1 - Ft,\ H_c^1(X, \mathscr{G})\big)}{\det(1 - Ft, H_c^2(X, \mathscr{G}))} \tag{5}$$

   by the Grothendieck trace formula.
2. We want to look at $H_c^2(X, \mathscr{G}) = H_c^2(U, \mathscr{F})$. By Poincaré duality, this is $H^0(U, \mathscr{F}^\vee(1))^\vee$ where (1) is a Tate twist. Taking the Tate twist out, we get

$$H_c^2(U, \mathscr{F}) = H^0(U, \mathscr{F}^\vee(1))^\vee = \Big((\mathscr{F}_{\overline{x}}^\vee)^{\pi_1(U, \overline{x})}\Big)^\vee (-1) = (\mathscr{F}_{\overline{x}})_{\pi_1(U, \overline{x})}(-1),$$

   where dualizing changes invariants to coinvariants.
   Now using the right-hand side of (5), we see that the poles of the L-function are of the form $\frac{1}{\alpha q}$ where α is an eigenvalue of $F_x \circlearrowleft (\mathscr{F}_{0\overline{x}})_{\pi_1(U, \overline{x})}$. This lifts to give an eigenvalue $\alpha^{d(x)}$ on $\mathscr{F}_{0\overline{x}}$ (this goes back to something Tyler said last time: Frobenius induces dth powers on the coinvariants). This implies $|\tau(\alpha)| \leq q^{\beta/2}$ by the previous Lemma, and so $\left|\frac{1}{\tau(\alpha q)}\right| > q^{-\beta/2-1}$, i.e., we have no poles inside a bounded disk.
3. We use the "factorization" (5) from before, where by the previous Lemma, $\tau L(U_0, \mathscr{G}_0, t)$ has no zeros or poles for $|t| \leq q^{-\beta/2-1}$, and $\tau L(X_0, \mathscr{G}_0, t)$ has no poles for $|t| \leq q^{-\beta/2-1}$. This implies the factor

$$\prod_{s \in |S_0|} \tau\Big(\det\Big(1 - t^{d(s)} F_s,\ \mathscr{G}_{0\overline{s}}\Big)\Big)^{-1}$$

   has no poles, either. Thus the eigenvalues of $F_s \colon \mathscr{G}_{0\overline{s}} \to \mathscr{G}_{0\overline{s}}$ are bounded: $|\tau(\alpha)| \leq q^{-\beta/2-1}$.
4. Play the same game with $j_{0*}(\mathscr{F}_0^{\otimes k})$ to get a bound $|\tau(\alpha)| \leq q^{-\beta/2-1/k}$ and take $k \to \infty$. ☐

Lemma 3.9. *If X_0 is a normal, irreducible algebraic scheme over k, and $\mathscr{G}_0$ is irreducible and smooth, and $j_0 \colon U_0 \hookrightarrow X_0$ where U_0 is a dense open subscheme of X_0, then $j_0^*(\mathscr{G}_0)$ is also irreducible.*

Proof. $\pi_1(U_0, \overline{a}) \twoheadrightarrow \pi_1(X_0, \overline{a})$. Recall from representation theory that if $G \to G/H \to \mathrm{GL}(V)$ is such that the second arrow is an irreducible representation, then the compositum is also an irreducible representation. ☐

The point of all this is to prove the following

Theorem 3.10 (Semicontinuity)**.** *Let $\mathscr{G}_0$ be a smooth sheaf on X_0 and let $j_0 \colon U_0 \hookrightarrow X_0$ be an open dense subscheme. Then,*

1. *$w(\mathscr{G}_0) = w(j_0^*(\mathscr{G}_0))$.*
2. *If $j_0^*(\mathscr{G}_0)$ is τ-pure of weight β, then $\mathscr{G}_0$ is τ-pure of weight β.*
3. *Let X_0 be irreducible and normal, and let $\mathscr{G}_0$ be irreducible. If $j_0^*(\mathscr{G}_0)$ is τ-mixed, then $\mathscr{G}_0$ is τ-pure.*
4. *Let X_0 be connected, let $j_0^*(\mathscr{G}_0)$ be τ-mixed, and let $\mathscr{G}_0$ be τ-pure of weight β at a single point $x \in |X_0|$ (that is, the condition for τ-purity holds only for the eigenvalues of Frobenius on the stalk at x). Then, $\mathscr{G}_0$ is τ-pure of weight β.*

Proof.

1. Assume X_0 is irreducible, and replace X by the normalization of $X_{0\text{red}}$. Then, if $\dim(X_0) = 1$, Lemma 3.8 from before applies, and we are done. If $\dim(X_0) > 1$, then connect any point $s \in |X_0 \setminus U_0|$ to a point in U_0 with a curve and apply Lemma 3.8 again (you need smoothness of $\mathscr{G}_0$ to say $H^0_S(X, \mathscr{G}) = 0$).
2. Apply (1) to $\mathscr{G}_0$ and $\mathscr{G}_0^\vee$ to get an upper and lower bound.
3. Apply Lemma 3.9: $j_0^*(\mathscr{G}_0)$ is irreducible, and so $j_0^*(\mathscr{G}_0)$ is pure; then, apply (2). Note that we possibly have to shrink U_0 to make sure $j_0^*(\mathscr{G}_0)$ has a filtration by smooth subsheaves.
4. Assume X_0 is irreducible, normal, and $\mathscr{G}_0$ is irreducible (work with each irreducible component and irreducible constituent of $\mathscr{G}_0$). This implies $j_0^*(\mathscr{G}_0)$ is τ-pure by (3), and so $\mathscr{G}_0$ is τ-pure by (2), and the weight obviously is β. ☐

We end with some definitions that will be useful later.

Definition 3.11. Let $\mathscr{G}_0$ be a sheaf over X_0. Define

$$w_{\text{gen}}(\mathscr{G}_0) = w(j_0^*(\mathscr{G}_0))$$

for any open dense subscheme $U_0 \overset{j_0}{\hookrightarrow} X_0$ on which $j_0^*(\mathscr{G}_0)$ is smooth.

Note that (1) in the previous Theorem implies this definition does not depend on the open set U_0 chosen.

Definition 3.12. We say $\mathscr{G}_0$ is *τ-real* if the characteristic polynomial $\tau \det(1 - F_x t,\ \mathscr{G}_{0\overline{x}}) \in \mathbf{R}[t]$ for all $x \in |X_0|$.

Remark 3.13. This has the obvious permanence properties.

We end with a result showing we can reduce to the τ-real case.

Lemma 3.14. *Let $\mathscr{G}_0$ be a smooth sheaf, τ-pure of weight β. Then, $\mathscr{G}_0$ is a direct summand of a τ-real and τ-pure sheaf of weight β.*

Proof. We would want to take $\mathscr{G}_0^\vee \oplus \mathscr{G}_0$, but this changes the weight; instead, we use $\left(\mathscr{G}_0^\vee \otimes \mathscr{L}_{\tau^{-1}(q^\beta)}\right) \oplus \mathscr{G}_0$. ☐

4 May 23—Weights II (Takumi Murayama)

Notation 4.1. As usual, $\kappa = \mathbf{F}_q$, $k = \overline{\kappa}$, X_0 is an algebraic (i.e., finite type) scheme over κ, $\mathscr{G}_0$ is a (Weil) sheaf on X_0, and $\tau\colon \overline{\mathbf{Q}} \xrightarrow{\sim} \mathbf{C}$ always denotes some isomorphism. If $x \in |X_0|$, we also denoted $d(x) = [\kappa(x) : \kappa]$ and $N(x) = \#\kappa(x) = q^{d(x)}$.

Recall from last time:

Definition 4.2. A *weight* of $\mathscr{G}_0$ is the quantity

$$\frac{\log\bigl(|\tau(\alpha)|^2\bigr)}{\log(N(x))}$$

where α is an eigenvalue of the Frobenius morphism $F_x\colon \mathscr{G}_{0\overline{x}} \to \mathscr{G}_{0\overline{x}}$ at a geometric point $\overline{x}$ lying over some $x \in |X_0|$. The *maximal weight* of $\mathscr{G}_0$ is

$$w(\mathscr{G}_0) = \sup_{x \in |X_0|} \sup_{\substack{\alpha \\ \text{eigenvalue} \\ F_x\colon \mathscr{G}_{0\overline{x}} \to \mathscr{G}_{0\overline{x}}}} \frac{\log\bigl(|\tau(\alpha)|^2\bigr)}{\log(N(x))}$$

if $\mathscr{G}_0 \neq 0$, and $-\infty$ otherwise.

Recall from last time (Lemma 3.7) that the maximal weight governs where the L-function converges:

Lemma 4.3. *The L-function*

$$\tau L(X_0, \mathscr{G}_0, t) = \prod_{x \in |X_0|} \tau \det\Big(1 - t^{d(x)} F_x,\ \mathscr{G}_{0\overline{x}}\Big)^{-1}$$

converges on the ball $|t| < q^{-w(\mathscr{G}_0)/2 - \dim(X_0)}$, *and has no zeros or poles in this ball.*

The goal of today's talk is to give an alternate description for what the maximal weight $w(\mathscr{G}_0)$ is, at least in the case when $\mathscr{G}_0$ is a τ-mixed sheaf on a smooth curve. It turns out that $w(\mathscr{G}_0)$ determines the radius of convergence of a certain power series we introduce later.

Notation 4.4. We introduce some new notation: for every $n \in \mathbf{Z}_{>0}$, we let $\kappa_n = \mathbf{F}_{q^n}$ denote the unique degree n extension of κ in k. $F_n \in \mathrm{Gal}(k/\kappa_n)$ denotes the geometric Frobenius over κ_n. In this case, we can describe the set of κ_n valued points as

$$X_0(\kappa_n) = \mathrm{Hom}_{\operatorname{Spec}\kappa}(\operatorname{Spec}\kappa_n, X_0) = X_0(k)^{F_n}$$

where $X_0(k) = \mathrm{Hom}_{\operatorname{Spec}\kappa}(\operatorname{Spec} k, X_0)$ are the k-valued (geometric) points of X_0.

Definition 4.5. The key definition for today is the following function:

$$f^{\mathscr{G}_0} = f_n^{\mathscr{G}_0} \colon \begin{cases} X_0(\kappa_n) \longrightarrow \mathbf{C} \\ \qquad \overline{x} \longmapsto \tau \operatorname{Tr}\Big(F_x^{n/d(x)},\ \mathscr{G}_{0\overline{x}}\Big) = \tau \operatorname{Tr}(F_n,\ \mathscr{G}_{0\overline{x}}) \end{cases}$$

where $x \in |X_0|$ is a closed point, and $\overline{x} \in X_0(\kappa_n)$ is a geometric point lying over it:

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[d] \arrow[drr, dashed, "\overline{x}"] & & \\
\operatorname{Spec}\kappa_n \arrow[rr, "x"] \arrow[dr] & & X_0 \arrow[dl] \\
& \operatorname{Spec}\kappa &
\end{tikzcd}$$

We won't denote the subscript n; hopefully it's not too confusing.

This forms a part of sheaf-function correspondence alluded to by Bhargav.

Definition 4.6. For any functions $f, g\colon X_0(\kappa_n) \to \mathbf{C}$, we define a scalar product and its associated norm:

$$(f, g)_n = \sum_{y \in X_0(\kappa_n)} f(y)\overline{g(y)}, \qquad \|f\|_n^2 = (f, f)_n.$$

Note that for the first sum to make sense, we think of $X_0(\kappa_n)$ as living inside $X_0(k)$.

These will give the space of functions $X_0(\kappa_n) \to \mathbf{C}$ an "L^2-structure," which will be used to define the Fourier transform.

Now recall from last time that Brandon rewrote the logarithmic derivative of the L-function as follows:

$$\frac{\tau L'(X_0, \mathscr{G}_0, t)}{\tau L(X_0, \mathscr{G}_0, t)} = \sum_{x \in |X_0|} \sum_{n=1}^{\infty} d(x) \tau \operatorname{Tr}(F_x^n) t^{d(x)n - 1} = \sum_{n=1}^{\infty} \Bigg(\sum_{\substack{x \in |X_0| \\ d(x) | n}} d(x) \tau \operatorname{Tr}\Big(F_x^{n/d(x)}\Big) \Bigg) t^{n-1}.$$

Now since

$$(f^{\mathscr{G}_0}, 1)_n = \sum_{y \in X_0(\kappa_n)} f^{\mathscr{G}_0}(y) = \sum_{y \in X_0(\kappa_n)} \tau \operatorname{Tr}\Big(F_x^{n/d(x)}\Big) = \sum_{\substack{x \in |X_0| \\ d(x) | n}} d(x) \tau \operatorname{Tr}\Big(F_x^{n/d(x)}\Big)$$

by using that a closed point in x with $d(x) \mid n$ corresponds to a $\mathrm{Gal}(\kappa_n/\kappa)$-orbit in $X_0(\kappa_n)$, we can rewrite the logarithmic derivative of the L-function as

$$\frac{\tau L'(X_0, \mathscr{G}_0, t)}{\tau L(X_0, \mathscr{G}_0, t)} = \sum_{n=1}^{\infty} (f^{\mathscr{G}_0}, 1)_n t^{n-1}.$$

We will today look at a similar power series, except with the "L^2-norm" $\|f^{\mathscr{G}_0}\|_n^2$ replacing $(f^{\mathscr{G}_0}, 1)_n$:

Definition 4.7. We define

$$\phi^{\mathscr{G}_0}(t) = \sum_{t=1}^{\infty} \|f^{\mathscr{G}_0}\|_n^2 \cdot t^{n-1} = \sum_{n=1}^{\infty} \left(\sum_{\overline{x} \in X_0(\kappa_n)} \left| \tau \operatorname{Tr}\left(F_x^{n/d(x)}\right) \right|^2 \right) \cdot t^{n-1}.$$

A possible reason for introducing $\phi^{\mathscr{G}_0}(t)$ is because it might work better with the Fourier transform, which will come later. As a bonus, its coefficients are nonnegative, and so it'll be easier to compute its radius of convergence.

We want to show properties of $\phi^{\mathscr{G}_0}(t)$ as we did for the L-function last time:

Lemma 4.8. *There is a constant C independent from n such that*

$$\|f^{\mathscr{G}_0}\|_n^2 \le C \cdot q^{n \cdot (w(\mathscr{G}_0) + \dim(X_0))}$$

for all $n \in \mathbf{Z}_{>0}$, so $\phi^{\mathscr{G}_0}(t)$ converges for $|t| < q^{-w(\mathscr{G}_0) - \dim(X_0)}$.

Proof. We proceed as in Lemma 3.7. First,

$$|f^{\mathscr{G}_0}(x)|^2 = \left| \tau \operatorname{Tr}\left(F_x^{n/d(x)}\right) \right|^2 \le r^2 \cdot q^{n \cdot w(\mathscr{G}_0)} \quad \text{where} \quad r = \max_{x \in |X_0|} \dim_{\overline{\mathbf{Q}}_\ell} \mathscr{G}_{0\overline{x}}$$

which we see is independent of x, and so

$$\|f^{\mathscr{G}_0}\|_n^2 = \sum_{x \in X_0(\kappa_n)} |f^{\mathscr{G}_0}(x)|^2 \le \# X_0(\kappa_n) \cdot r^2 \cdot q^{n \cdot w(\mathscr{G}_0)} \le C \cdot q^{n \cdot (w(\mathscr{G}_0) + \dim(X_0))},$$

where as before, the last inequality is by the Noether normalization theorem. ☐

This tells us a lower bound for the radius of convergence of $\phi^{\mathscr{G}_0}(t)$, but begs the question of whether $q^{-w(\mathscr{G}_0) - \dim(X_0)}$ is *exactly* the radius of convergence. Our main result is that this in fact *is* the radius of convergence, in some very nice cases.

Before we state the result, we introduce one more piece of notation:

Definition 4.9. We define the *L^2-norm* of a sheaf $\mathscr{G}_0$ as

$$\|\mathscr{G}_0\| = \sup \left\{ \rho \;\middle|\; \limsup_n \frac{\|f^{\mathscr{G}_0}\|_n^2}{q^{n(\rho + \dim(X_0))}} > 0 \right\}.$$

Key Observation 4.10. $q^{-\|\mathscr{G}_0\| - \dim(X_0)}$ is the radius of convergence of $\phi^{\mathscr{G}_0}(t)$.

Note that by our discussion of the radius of convergence above, we always have

$$\|\mathscr{G}_0\| \le w(\mathscr{G}_0).$$

The content of the following theorem is that we sometimes get the opposite inequality.

Recall that the *generic maximal weight* $w_{\mathrm{gen}}(\mathscr{G}_0) = w(j_0^* \mathscr{G}_0)$ where $j_0 \colon U_0 \hookrightarrow X_0$ is any open dense immersion of a smooth subscheme. This was well-defined by the theorem on semicontinuity of weights from last time.

Theorem 4.11 (Radius of Convergence). *Let $\mathscr{G}_0$ be a τ-mixed sheaf on an algebraic scheme X_0 of dimension $\dim X_0 \le 1$. Let $j_0 \colon U_0 \hookrightarrow X_0$ be the open subscheme of X_0 consisting of all irreducible components with dimension $= \dim X_0$. Then we have (with the convention $w_{gen}(j_0^*(\mathscr{G}_0)) = -\infty$ for $U_0 = \emptyset$)*

1. $\|\mathscr{G}_0\| = \max(w_{gen}(j_0^*(\mathscr{G}_0)),\ w(\mathscr{G}_0) - 1)$
2. *Assume* X_0 *to be a smooth curve. If* $H^0_E(X, \mathscr{G}) = 0$ *for all closed subsets* E *of* X*, then*

$$\|\mathscr{G}_0\| = w(\mathscr{G}_0).$$

Remark 4.12*.* Here, a *curve* means a one-dimensional scheme X_0 that has pure dimension 1.

Proof. First, (1) ⇒ (2) follows since if X_0 is smooth, we have

$$\|\mathscr{G}_0\| = \max(w_{\text{gen}}(j_0^*(\mathscr{G}_0)), w(\mathscr{G}_0) - 1) = \max(w(\mathscr{G}_0), w(\mathscr{G}_0) - 1) = w(\mathscr{G}_0).$$

The proof of (1) will boil down to computing the radius of convergence of the power series $\phi^{\mathscr{G}_0}(t)$.

As always, we can assume X_0 is reduced. We moreover claim that it suffices to consider when X_0 is connected. First, write

$$\phi^{\mathscr{G}_0}(t) = \sum_{\substack{W \subset X_0 \\ \text{connected component}}} \phi^{\mathscr{G}_0|_W}(t).$$

Since each $\phi^{\mathscr{G}_0|_W}(t)$ has non-negative coefficients, we have that the radius of convergence of $\phi^{\mathscr{G}_0}(t)$ is the minimum of the radii of convergence of $\phi^{\mathscr{G}_0|_W}(t)$, and so

$$\|\mathscr{G}_0\| = \max_{\substack{W \subset X_0 \\ \text{connected component}}} \left\{ \|\mathscr{G}_0|_W\| \right\},$$

and (1) would follow from the connected case.

The rest of the proof is by some case work. We first prove the $\dim X_0 = 0$ case, then prove the case where $\mathscr{G}_0$ is a *smooth* and τ*-pure* sheaf on a smooth affine curve X_0, then the same for τ-mixed $\mathscr{G}_0$, and then finally the general case.

Note that if X_0 is smooth, then (1) just says $\|\mathscr{G}_0\| = w(\mathscr{G}_0)$. This is what we will show in the first three cases. Also, since we already have shown $\|\mathscr{G}_0\| \le w(\mathscr{G}_0)$, it suffices to show the opposite inequality.

Case 1. $\dim X_0 = 0$.

Assume X_0 is connected as above, and let $s \in |X_0|$ be the unique point in X_0. Then, consider the stalk $V = \mathscr{G}_{0\bar{s}}$ of $\mathscr{G}_0$ as a **C**-vector space via the isomorphism $\tau\colon \overline{\mathbf{Q}}_\ell \xrightarrow{\sim} \mathbf{C}$. The Frobenius map $F_s\colon V \to V$ can then be represented by a complex matrix A with entries in **C**. Let $\overline{A}$ be the conjugate matrix. Then, the function

$$\det\Bigl(\mathbf{1}_V - A \otimes \overline{A} \cdot t^{d(s)}\Bigr)^{-1}$$

has logarithmic derivative

$$\sum_{n=1}^{\infty} d(s)\operatorname{Tr}\bigl((A \otimes \overline{A})^n\bigr)\, t^{d(s)n-1} = \sum_{n=1}^{\infty} d(s)|\operatorname{Tr}(A^n)|^2 t^{d(s)n-1} = \sum_{n=1}^{\infty} \|f^{\mathscr{G}_0}\|_n^2 \cdot t^{n-1} = \phi^{\mathscr{G}_0},$$

and so the radius of convergence for $\phi^{\mathscr{G}_0}(t)$ is at most

$$\min_{\substack{\alpha,\beta \\ \text{eigenvalues} \\ F_s\colon \mathscr{G}_{0\bar{s}} \to \mathscr{G}_{0\bar{s}}}} |\tau(\alpha)\tau(\beta)|^{-1/d(s)} = \min_{\substack{\alpha \\ \text{eigenvalues} \\ F_s\colon \mathscr{G}_{0\bar{s}} \to \mathscr{G}_{0\bar{s}}}} |\tau(\alpha)|^{-2/d(s)} = q^{-w(\mathscr{G}_0)}$$

since these are the values of t for which $\det\bigl(\mathbf{1}_V - A \otimes \overline{A} \cdot t^{d(s)}\bigr)^{-1}$ diverges. Thus, $\|\mathscr{G}_0\| \ge w(\mathscr{G}_0)$, and we conclude $\|\mathscr{G}_0\| = w(\mathscr{G}_0)$.

Case 2. $\mathscr{G}_0$ a smooth, τ-pure sheaf of weight β on a smooth affine curve X_0.

Assume X_0 is connected as before, and also assume X_0 is geometrically irreducible like in the proof of the "curve case" from last time. Assume also that $\mathscr{G}_0 \neq 0$, for otherwise there would be nothing to show. Now consider the "complex conjugate" sheaf

$$\overline{\mathscr{G}}_0 = \mathscr{G}_0^\vee \otimes \mathscr{L}_{\tau^{-1}(q^\beta)}$$

introduced last time to show $\mathscr{G}_0$ is the direct summand of a real sheaf. Then, $\mathscr{G}_0 \otimes \overline{\mathscr{G}_0}$ is τ-real, and the power series $\phi^{\mathscr{G}_0}(t)$ is the logarithmic derivative of the L-series

$$\begin{aligned}\tau L(X_0, \mathscr{G}_0 \otimes \overline{\mathscr{G}}_0, t) &= \prod_{x \in |X_0|} \tau \det\Big(1 - F_x t^{d(x)},\ \mathscr{G}_{0\overline{x}} \otimes \overline{\mathscr{G}}_{0\overline{x}}\Big)^{-1} \\ &= \frac{\tau \det\big(1 - Ft,\ H^1_c(X, \mathscr{G} \otimes \overline{\mathscr{G}})\big)}{\tau \det\big(1 - Ft,\ H^2_c(X, \mathscr{G} \otimes \overline{\mathscr{G}})\big)},\end{aligned}$$

where the last equality is by the Grothendieck trace formula, since

$$(f^{\mathscr{G}_0 \otimes \overline{\mathscr{G}}_0}(\overline{x}), 1)_n = |f^{\mathscr{G}_0}(\overline{x})|^2_n.$$

To show that $\phi^{\mathscr{G}_0}$ has the correct radius of convergence, we will use that $\mathscr{G}_0 \otimes \overline{\mathscr{G}}_0$ is τ-pure of weight 2β by assumption on τ-purity.

1. Lemma 3.7 says that this L-function has no zeros or poles in the region $|t| < q^{-\beta-1}$.
2. We now want to show where poles *could* live. Using the last expression as a rational function, the proof of the "curve case" of semicontinuity from last time shows the poles of the L-function written above are of the form $\frac{1}{\alpha q}$ where α is an eigenvalue of

   $$F_x \circlearrowright (\mathscr{G}_{0\overline{x}} \otimes \overline{\mathscr{G}}_{0\overline{x}})_{\pi(X,\overline{x})}$$

   for some stalk. As before, $\alpha^{d(x)}$ is then an eigenvalue of $F_x \circlearrowright \mathscr{G}_{0\overline{x}} \otimes \overline{\mathscr{G}}_{0\overline{x}}$, hence has

   $$|\tau\alpha^{d(x)}|^2 = q^{2d(x)\beta}$$

   by assumption on τ-purity. We therefore have

   $$\left|\tau\left(\frac{1}{\alpha q}\right)\right| = q^{-\beta-1},$$

   and so any pole of $\tau L(X_0, \mathscr{G}_0 \otimes \overline{\mathscr{G}}_0, t)$ has norm $q^{-\beta-1}$.
3. Each "local L-factor"

   $$\tau \det\Big(1 - F_x t^{d(x)},\ \mathscr{G}_{0\overline{x}} \otimes \overline{\mathscr{G}}_{0\overline{x}}\Big)^{-1}$$

   is a power series with leading coefficient 1. First of all, since $\mathscr{G}_0 \neq 0$, each local L-factor has some poles, Next, by Lemma 3.7, each local L-factor has a radius of convergence at least $q^{-w(\mathscr{G}_0)} = q^{-\beta}$, and their poles have norm $q^{-\beta-1}$ by (2), and so each local L-factor has a radius of convergence $q^{-\beta-1}$. Thus, their product has a radius of convergence $\leq q^{-\beta-1}$ (this uses some general facts about power series [KW01, Remark 2.17]).

This shows the radius of convergence of $\phi^{\mathscr{G}_0}$ is $q^{-\beta-1} = q^{-\|\mathscr{G}_0\|-1}$, and so $\|\mathscr{G}_0\| = w(\mathscr{G}_0)$.

Case 3. $\mathscr{G}_0$ a smooth, τ-mixed sheaf on a smooth affine curve X_0.

Consider the filtration

$$0 = \mathscr{G}_0^{(0)} \subset \mathscr{G}_0^{(1)} \subset \cdots \subset \mathscr{G}_0^{(r)} = \mathscr{G}_0$$

where $\mathscr{G}_0^{(j)}/\mathscr{G}_0^{(j-1)}$ is τ-pure of weight β_j. We would want to just use how traces interact with filtrations, but the issue is that the coefficients of $\phi^{\mathscr{G}_0}(t)$ are a bit harder to work with because they are squares of things. Regardless, we can replace $\mathscr{G}_0$ by its "semisimplification"

$$\bigoplus_j \mathscr{G}_0^{(j)}/\mathscr{G}_0^{(j-1)} =: \mathscr{F}_0 \oplus \mathscr{H}_0,$$

where $\mathscr{F}_0$ is the direct sum of all summands that are τ-pure of weight $w(\mathscr{G}_0)$, since this will not change the traces involved in the definition of $\phi^{\mathscr{G}_0}(t)$. Note $w(\mathscr{H}_0) < w(\mathscr{F}_0)$ by assumption.

Now since $f^{\mathscr{G}_0} = f^{\mathscr{F}_0} + f^{\mathscr{H}_0}$ implies $\|f^{\mathscr{G}_0}\|^2_n = \|f^{\mathscr{F}_0}\|^2_n + 2\operatorname{Re}(f^{\mathscr{F}_0}, f^{\mathscr{H}_0})_n + \|f^{\mathscr{H}_0}\|^2_n$, we obtain

$$\phi^{\mathscr{G}_0}(t) = \phi^{\mathscr{F}_0}(t) + \sum_{n=1}^{\infty} 2\operatorname{Re}(f^{\mathscr{F}_0}, f^{\mathscr{H}_0})_n t^{n-1} + \sum_{n=1}^{\infty} \|f^{\mathscr{H}_0}\|^2_n t^{n-1}.$$

1. The first term has radius of convergence $q^{-w(\mathscr{G}_0)-1}$.
2. Lemma 4.8 implies the last term has radius of convergence at least $q^{-w(\mathscr{H}_0)-1}$.
3. The inequality
$$\left|2\mathrm{Re}(f^{\mathscr{F}_0}, f^{\mathscr{H}_0})_n\right| \le 2\|f^{\mathscr{F}_0}\|_n \cdot \|f^{\mathscr{H}_0}\|_n \le C \cdot q^{n\cdot\left(\frac{w(\mathscr{G}_0)+w(\mathscr{H}_0)}{2}+1\right)}$$
again from Lemma 4.8 implies the second term has radius of convergence at least $q^{-\left(\frac{w(\mathscr{G}_0)+w(\mathscr{H}_0)}{2}+1\right)}$.

Thus, $\phi^{\mathscr{G}_0}(t)$ has radius of convergence $q^{-w(\mathscr{G}_0)-1}$, so $w(\mathscr{G}_0) = \|\mathscr{G}_0\|$ as required.

Case 4. The general case.

This is the first time the term $w_{\mathrm{gen}}(j_0^*(\mathscr{G}_0))$ in the maximum will appear.

Recall X_0 can be assumed to be reduced. In this case, we can find an open affine smooth curve (note curve means of pure dimension 1)
$$h_0\colon V_0 \hookrightarrow X_0$$
such that the complement
$$i_0\colon S_0 \hookrightarrow X_0$$
of V_0 is finite and such that $h_0^*(\mathscr{G}_0) = \mathscr{F}_0$ is smooth on V_0. Put also $\mathscr{H}_0 = i_0^*(\mathscr{G}_0)$.

Now consider $j_0\colon U_0 \hookrightarrow X_0$ and the sheaf $j_0^*(\mathscr{G}_0)$ on U_0. From definition of U_0, we have $V_0 \subset U_0$ and
$$w_{\mathrm{gen}}(j_0^*(\mathscr{G}_0)) = w(\mathscr{F}_0)$$
by the semicontinuity theorem from last time. Then,
$$\begin{aligned}\max\{w(\mathscr{F}_0), w(\mathscr{H}_0)\} &= w(\mathscr{G}_0),\\ \max\{w(\mathscr{F}_0), w(\mathscr{H}_0)-1\} &= \max\{w_{\mathrm{gen}}(j_0^*(\mathscr{G}_0)), w(\mathscr{G}_0)-1\}\,.\end{aligned}$$
where the first equality is by definition of maximal weights, and the second is by considering how $w(\mathscr{F}_0)$ and $w(\mathscr{H}_0)-1$ compare.

Now use the short exact sequence
$$0 \longrightarrow h_{0!}(\mathscr{F}_0) \longrightarrow \mathscr{G}_0 \longrightarrow i_{0*}(\mathscr{H}_0) \longrightarrow 0$$
to obtain
$$\phi^{\mathscr{G}_0}(t) = \phi^{\mathscr{F}_0}(t) + \phi^{\mathscr{H}_0}(t)$$
by considering stalks. Now the coefficients of these power series are nonnegative so the radius of convergence of $\phi^{\mathscr{G}_0}$ is the minimum of the radii of convergence of $\phi^{\mathscr{F}_0}$ and $\phi^{\mathscr{H}_0}$, which are $q^{-w(\mathscr{F}_0)-1}$ and $q^{-w(\mathscr{H}_0)}$, respectively. Thus,
$$\|\mathscr{G}_0\| = \max\{w(\mathscr{F}_0), w(\mathscr{H}_0)-1\} = \max\{w_{\mathrm{gen}}(j_0^*(\mathscr{G}_0)), w(\mathscr{G}_0)-1\}\,. \qquad \square$$

5 May 25—Monodromy (Emanuel Reinecke)

Notation 5.1. X_0 denotes a geometrically connected, normal scheme of finite type over κ; $\overline{x}$ denotes a geometric point; k denotes the algebraic closure of κ; X denotes the base change of X_0 to $\overline{\kappa}$; and $\tau\colon \overline{\mathbf{Q}}_l \xrightarrow{\sim} \mathbf{C}$ is a fixed isomorphism.

Last two times, we defined pure and mixed sheaves, and weights on them. The problem with this, however, is that the definition for weights so far only make sense for mixed sheaves, and we don't know how to define them for general sheaves.

Today's goal is to introduce a new notion of weights called "determinant weights." These will be defined for all lisse sheaves, and will recover the standard notion for τ-real sheaves (which we recall are sheaves whose characteristic polynomials of F_a are real polynomials for all closed points $a \in |X_0|$).

The advantage of using determinant weights is that they are fairly easy to compute, and behave well with the usual cadre of operations (pullback, tensor product, exterior powers…).

We remind the reader of the following diagram from Tyler's lecture, which we referred to as the monodromy exact sequence:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X,\overline{x}) \arrow[r] \arrow[d, equal] & \pi_1(X_0,\overline{x}) \arrow[r] \arrow[d, phantom, "\cup"] & \mathrm{Gal}(k/\kappa) \cong \hat{\mathbf{Z}} \arrow[r] \arrow[d, phantom, "\cup"] & 1 \\
1 \arrow[r] & \pi_1(X,\overline{x}) \arrow[r] & W(X_0,\overline{x}) \arrow[r] & W(k/\kappa) \cong \mathbf{Z} \arrow[r] & 1
\end{tikzcd} \tag{6}$$

Recall also that a lisse (Weil) sheaf $\mathscr{G}_0$ corresponds by Tannakian duality to a finite dimensional representation of $W(X_0,\overline{x})$; thus, a rank 1 lisse sheaf corresponds to a character of $W(X_0,\overline{x})$.

5.1 Determinant Weights

5.1.1 Rank 1 lisse sheaves

We first consider rank 1 (Weil) sheaves, which we can describe explicitly: they are all τ-pure.

Let $\mathscr{G}_0$ be a lisse (smooth) sheaf of rank 1 on X_0. Let $\chi\colon W(X_0,\overline{x}) \to \overline{\mathbf{Q}}_\ell^*$ be the corresponding character. Our first main theorem is the following:

Theorem 5.2. *If $\chi\colon W(X_0,\overline{x}) \to \overline{\mathbf{Q}}_\ell^*$ is a continuous character, then the image of $\pi_1(X,\overline{x})$ via χ in $\overline{\mathbf{Q}}_\ell^*$ is finite.*

We can restate this result in the following ways:

Corollary 5.3. *There is a positive integer M such that $\chi^M|_{\pi_1(X,\overline{x})}$ is the trivial map.*

Corollary 5.4. *We can write $\chi = \chi_1 \cdot \chi_2$, where χ_1 is torsion, and we have a factorization*

$$\begin{tikzcd}
W(X_0,\overline{x}) \arrow[r, "\chi_2"] \arrow[dr] & \overline{\mathbf{Q}}_\ell^* \\
& W(k/\kappa) \arrow[u, dashed]
\end{tikzcd}$$

Proof. The short exact sequence in (6) splits, since we can choose an arbitrary preimage of the generator $\mathbf{Z}$ (note, however, that the splitting is not canonical). Thus, $W(X_0,\overline{x})$ is a semidirect product, and we have the required decomposition. ☐

Corollary 5.5. *$\mathscr{G}_0 \cong \mathscr{F}_0 \otimes \mathscr{L}_{\mathfrak{b}}$, where $\mathscr{F}_0$ is torsion and $\mathscr{L}_{\mathfrak{b}}$ is pulled back from κ. In particular, $\mathscr{G}_0$ is τ-pure.*

Proof. By Tannakian duality, Corollary 5.4 implies $\mathscr{G}_0 = \mathscr{F}_0 \otimes \mathscr{L}_{\mathfrak{b}}$ for some $\mathfrak{b} \in \overline{\mathbf{Q}}_\ell^*$, where

- $\mathscr{F}_0^{\otimes m} = \overline{\mathbf{Q}}_\ell$, hence all eigenvalues are roots of unity, and also $\mathscr{F}_0$ is pure of weight 0;
- $\mathscr{L}_{\mathfrak{b}}$ (either as a sheaf on $\operatorname{Spec}\kappa$ or X_0) has weight

$$\frac{\log(|\tau(\mathfrak{b})|)^2}{\log(q)}. \tag{7}$$

Thus, $\mathscr{G}_0$ is τ-pure of weight as in (7). ☐

Remark 5.6. Note that [KW01] demand that X_0 is geometrically irreducible, and in fact switch back and forth between assuming geometric irreducibility and geometric connectivity. But these notions are equivalent under our standing assumptions in Notation 5.1: κ is perfect, and X_0 normal, and so $X =: X_0 \otimes k$ is a base extension to a separable field extension, hence is still normal. Under these hypotheses, geometric connectivity implies geometric irreducibility.

We now give a sketch of the proof of Theorem 5.2.

Proof Sketch. By dévissage, we can reduce to the case where X_0 is a smooth, projective, geometrically connected curve (we will see this step later), and so we first consider this special case. We will show that we have a commutative diagram:

$$\begin{tikzcd}
\pi_1(X,\overline{x}) \arrow[r] \arrow[d] & W(X_0,\overline{x}) \arrow[r, "\chi"] \arrow[d] & \overline{\mathbf{Q}}_\ell^* \\
\mathrm{Pic}^0(X_0)(\kappa) \arrow[r] & W(X_0,\overline{x})_{\mathrm{ab}} \arrow[ur, dashed] &
\end{tikzcd} \tag{8}$$

Since $\mathrm{Pic}^0(X_0)(\kappa)$ has only finitely many points (since $\mathrm{Pic}^0(X_0) = \mathrm{Jac}(X_0)$ is a projective variety), this would imply the Theorem.

Let $K = K(X_0)$ be the function field of X_0. Then, we have a surjection

$$G_K = \pi_1(K) \twoheadrightarrow \pi_1(X_0, \overline{x}) = G_K^{\mathrm{ur}},$$

where the latter group is the Galois group of the maximal Galois extension of K that is everywhere unramified; since the pullback of a connected finite étale cover is connected, the connectivity criterion implies that the map above is surjective.

We now look at the abelianization of the diagram (6), and describe the elements of the top row below using class field theory for function fields:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Pic}^0(X_0)(\kappa) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & \mathrm{Pic}(X_0)(\kappa) \arrow[d, phantom, sloped, "\cong"] \arrow[dr, bend left, "D\mapsto q^{-\deg D}"] & & \\
0 \arrow[r] & K^*\Big\backslash (\mathbf{A}_K^*)^1 \Big/ \prod_v \mathcal{O}_v^* \arrow[r] \arrow[d, "\wr"'] & K^*\Big\backslash \mathbf{A}_K^* \Big/ \prod_v \mathcal{O}_v^* \arrow[r, "\prod_v \|\cdot\|_v"] \arrow[d, "\psi"', "\wr"] & q^{\mathbf{Z}} \arrow[r] \arrow[d, "q\mapsto F"] & 0 \\
0 \arrow[r] & I_K \arrow[r] & W(X_0,\overline{x})_{\mathrm{ab}} \arrow[r, "\deg"] \arrow[d, phantom, "\cap"] & W(k/\kappa) \cong \mathbf{Z} \arrow[r] \arrow[d, phantom, "\cap"] & 0 \\
 & \pi_1(X,\overline{x})_{\mathrm{ab}} \arrow[r] \arrow[u, two heads] & \pi_1(X_0,\overline{x})_{\mathrm{ab}} \arrow[r] & \mathrm{Gal}(k/\kappa) \cong \hat{\mathbf{Z}} \arrow[r] & 0
\end{tikzcd}$$

Note $\pi_1(X_0, \overline{x})_{\mathrm{ab}} = G_K^{\mathrm{ur,ab}}$, the Galois group of the maximal everywhere unramified *abelian* extension of $K(X_0)$, and I_K is the kernel of the degree map $W(X_0, \overline{x})_{\mathrm{ab}} \to W(k/\kappa) \cong \mathbf{Z}$. Also, $(\mathbf{A}_K^*)^1$ denotes the elements of $\mathbf{A}_K^*$ with norm 1. By commutativity, the commutative diagram (8) follows.

We now give an idea for how to perform the dévissage reduction alluded to at the beginning of our proof. Assume for simplicity that $X_0 \subset \mathbf{P}^N$ is smooth, projective, geometrically connected, and of dimension ≥ 2; note that the projectivity can be assumed by a theorem of Grothendieck, which says that replacing $\mathscr{G}_0$ by $\mathscr{G}_0^M$ for some $M > 0$, we can assume that $\mathscr{G}_0$ extends to the projective closure of (a suitable open subset of) X. Then, Bertini's theorem implies that for a general linear subspace $L \subset \mathbf{P}^N$ of codimension $(\dim X - 1)$, the intersection $C \coloneqq X \cap L$ is a smooth irreducible curve. After base field extension (which is okay since we may replace χ by χ^m), we can assume L and C are defined over κ, and so $L = L_0 \otimes_\kappa k$ and $C = C_0 \otimes_\kappa k$. Assuming (after possible conjugation of $\pi_1(X, \overline{x})$) that $\overline{x} \in C$, we apply the connectivity criterion again to conclude $\pi_1(C, \overline{x}) \twoheadrightarrow \pi_1(X, \overline{x})$. $\square$

Remark 5.7. Theorem 5.2 is not true if X is not normal. Take $\mathbf{P}^1_\kappa$ and glue 0 to ∞ to form a nodal cubic X. We can take a trivial sheaf on $\mathbf{P}^1_\kappa$ and glue the fibres over 0 and ∞ by any $\alpha \in \mathbf{Z}_\ell^*$. Then the image of the geometric fundamental group is $\alpha^{\hat{\mathbf{Z}}}$, which is usually infinite.

Remark 5.8. We have $I_K \cong \pi_1(X, \overline{x})_{\mathrm{ab}}/(F - \mathrm{id})\pi_1(X, \overline{x})_{\mathrm{ab}}$, and so we are using the fact that the Frobenius F does not have a 1-eigenspace on $\pi_1(X, \overline{x})_{\mathrm{ab}}$. Roughly speaking, we are using that $\pi_1(X, \overline{x})_{\mathrm{ab}}$ does not have a weight zero component.

5.1.2 Determinant weights in general

Now we consider sheaves $\mathscr{G}_0$ of higher rank. Consider a composition series

$$0 = \mathscr{G}_0^{(0)} \subset \mathscr{G}_0^{(1)} \subset \cdots \subset \mathscr{G}_0^{(r)} = \mathscr{G}_0,$$

where $\mathscr{F}_0^{(i)} := \mathscr{G}_0^{(i)}/\mathscr{G}_0^{(i-1)}$ are irreducible constituents of rank r_i. Note such a composition series exists since $\mathscr{G}_0$ corresponds by Tannakian duality to a finite dimensional representation. Then, Proposition 5.5 implies that $\bigwedge^{r_i} \mathscr{F}_0^{(i)}$ is τ-pure.

Definition 5.9. The *determinant weights* of $\mathscr{G}_0$ are

$$\frac{1}{r_i} \cdot w\Big(\bigwedge^{r_i} \mathscr{F}_0^{(i)}\Big).$$

5.2 Monodromy

5.2.1 Remarks on semidirect products

The material in this section was spread throughout the talk. We will be using basic properties of semidirect products, so we list them here. We omit proofs for the easy ones.

Let G and H be two groups. Recall that the outer isomorphism group $\operatorname{Out}(H)$ is the quotient of the automorphism group $\operatorname{Aut}(H)$ by the inner automorphisms.

Proposition 5.10. *Let $\alpha\colon G \to \operatorname{Aut}(H)$ and $\beta\colon G \to \operatorname{Aut}(H)$ be two actions of G on H. Then the two sequences*

$$1 \longrightarrow H \longrightarrow G \ltimes_\alpha H \longrightarrow G \longrightarrow 1$$

and

$$1 \longrightarrow H \longrightarrow G \ltimes_\beta H \longrightarrow G \longrightarrow 1$$

are isomorphic if and only if the composite maps $G \to \operatorname{Out}(H)$ are equal.

Proposition 5.11. *Let $\rho\colon G \ltimes H \to \operatorname{GL}(V)$ be a representation of $G \ltimes H$. Then, $G \ltimes H$ permutes the H-subrepresentations of V in a containment-preserving way.*

Proposition 5.12. *Let $\rho\colon G \ltimes H \to \operatorname{GL}(V)$ be a semisimple representation of $G \ltimes H$. Then $\rho|_H$ is semisimple.*

Proof. We immediately reduce to the case that ρ is a simple $G \ltimes H$ representation. An H-subrepresentation of V is simple and only if it has no proper H-subrepresentations, so $G \ltimes H$ preserves the set of simple H-subrepresentations.

Let W be the span of the simple H-subrepresentations. Then, W is a $G \ltimes H$ subrepresentation of V, and V is simple, so $V = W$. This shows that W is semi-simple as an H-representation. ☐

5.2.2 Group theory of monodromy

Once again, let $\mathscr{G}_0$ be a lisse sheaf on X_0. Let $\rho\colon W(X_0, \overline{x}) \to \operatorname{GL}(V)$ be the corresponding representation, where $V := \mathscr{G}_{0\overline{x}}$ is a $\overline{\mathbf{Q}}_\ell$-vector space, which is finite-dimensional since $\mathscr{G}_0$ is lisse. Note that by definition of a $\overline{\mathbf{Q}}_\ell$ sheaf, ρ is in fact defined over some finite extension E such that $\mathbf{Q}_\ell \subset E \subset \overline{\mathbf{Q}}_\ell$, and so $V = W \otimes_E \overline{\mathbf{Q}}_\ell$.

Definition 5.13. The images $\rho(\pi_1(X, \overline{x}))$ and $\rho(W(X_0, \overline{x}))$ in $\operatorname{GL}(W)$ are called the *geometric monodromy group* and the *arithmetic monodromy group*, respectively.

Recollections 5.14. Let G denote a group scheme defined over a field K, where $\operatorname{Char} K = 0$. Recall the following definitions and facts from representation theory:

1. G is a *linear algebraic group* if there exists a closed embedding $G \hookrightarrow \operatorname{GL}_n$; equivalently, G/K is of finite type and affine. G is automatically smooth since $\operatorname{Char} K = 0$.

   Examples 5.15. GL_n is trivially a linear algebraic group. A *torus* T/K (i.e., an algebraic group T such that $T_{\overline{K}} \simeq \mathbf{G}_m^r$ for some r) is also.

2. G is *(linearly) reductive* if $\mathsf{Rep}(G)$ is semisimple (this definition only works in characteristic zero). Alternatively (and this is the definition used in all characteristics), recall that $U \subset G$ is *unipotent* if there exists a closed embedding
$$U \hookrightarrow U_n = \left\{ (a_{ij}) \in \mathrm{GL}_n \;\middle|\; \begin{array}{l} a_{ii} = 1 \\ a_{ij} = 0 \text{ for all } i > j \end{array} \right\},$$
and then the *unipotent radical* $R_u(G)$ is defined to be the maximal connected, smooth, normal, unipotent subgroup of G. Then, one can show that G is reductive if and only if $R_u(G_{\overline{K}}) = \{1\}$.

   Examples 5.16. GL_n and tori T are reductive.

   Fact 5.17. If G is a connected, smooth, commutative linear algebraic group, then $G \cong U \times T$ where U is unipotent and T is a torus. In particular, if G is also reductive, then G is a torus.

   $R_u(-)$ commutes with finite field extensions $K \subset L$, that is, $R_u(G_L) = R_u(G)_L$.
3. The *radical* $R(G)$ of G is the maximal normal, smooth, connected, solvable subgroup of G. $R(-)$ commutes with finite field extensions $K \subset L$, that is, $R(G_L) = R(G)_L$. G is called *semisimple* if $R(G_{\overline{K}}) = \{1\}$. Observe that if G is semisimple, then it is reductive, since $R(G) \supset R_u(G)$.

   Example 5.18. If T is a torus, then $R(T_{\overline{K}}) = T_{\overline{K}}$.

   Fact 5.19. If G is a connected, reductive linear algebraic group over a perfect field K, then $R(G) = Z(G)^\circ_{\mathrm{red}}$, the maximal central torus, where H° denotes the identity component of H. In particular, G is semisimple if and only if $Z(G)$ is finite.
4. By Fact 5.17, if G is a reductive group, then the center $Z(G)$ is the product of a torus and a finite group, and we have a short exact sequence
$$1 \longrightarrow Z(G) \longrightarrow G \longrightarrow G_{\mathrm{adj}} \longrightarrow 1.$$
The same Fact shows that, again assuming G is reductive, the abelianization G_{ab} is the product of a torus and a finite group, and we have another short exact sequence:
$$1 \longrightarrow [G,G] \longrightarrow G \longrightarrow G_{\mathrm{ab}} \longrightarrow 1.$$
Here, the groups G_{adj} and $[G,G]$ are semisimple, and $[G,G] \to G_{\mathrm{adj}}$ is an isogeny. In particular, G is semisimple if and only if $Z(G)$ is finite (cf. Fact 5.19), if and only if G_{ab} is finite.

   Example 5.20. If $G = \mathrm{GL}_n$, then the first sequence is
$$1 \longrightarrow \mathbf{G}_m \longrightarrow \mathrm{GL}_n \longrightarrow \mathrm{PGL}_n \longrightarrow 1,$$
and the second is
$$1 \longrightarrow \mathrm{SL}_n \longrightarrow \mathrm{GL}_n \longrightarrow \mathbf{G}_m \longrightarrow 1.$$

We now return to our setting. Let $\mathscr{G}_0$ be a lisse sheaf, and denote $V = \mathscr{G}_{0\overline{x}}$ to be a stalk of $\mathscr{G}_0$ at a geometric point $\overline{x}$. Recall $\rho\colon W(X_0, \overline{x}) \to \mathrm{GL}(V)$ denotes the representation of $W(X_0, \overline{x})$ corresponding to $\mathscr{G}_0$ via Tannakian duality.

Notation 5.21. We denote G_{geom} to be the Zariski closure of $\rho(\pi_1(X, \overline{x}))$ in $\mathrm{GL}(V)$, that is, it is the smallest Zariski-closed subgroup of $\mathrm{GL}(V)$ whose $\overline{\mathbf{Q}}_\ell$-points contain $\rho(\pi_1(X, \overline{x}))$. G_{geom} is a linear algebraic group over $\overline{\mathbf{Q}}_\ell$.

We note from the proof of Corollary 5.4 that the short exact sequence
$$1 \longrightarrow \pi_1(X, \overline{x}) \longrightarrow W(X_0, \overline{x}) \longrightarrow W(k/\kappa) \cong \mathbf{Z} \longrightarrow 1$$

is split (non-canonically). In particular, we obtain an action of $\mathbf{Z}$ on $\pi_1(X,\overline{x})$, canonical up to inner automorphism. Applying ρ to the first two terms of this short exact sequence, we get a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X,\overline{x}) \arrow[r] \arrow[d] & W(X_0,\overline{x}) \arrow[r] \arrow[d, "\rho"] & W(k/\kappa) \arrow[r] & 1 \\
1 \arrow[r] & G_{\mathrm{geom}} \arrow[r, hook, "/" marking] & \mathrm{GL}(V) & &
\end{tikzcd}$$

by definition of G_{geom}.

Now choose some $g \in W(X_0,\overline{x})$ lifting $1 \in \mathbf{Z}$. Note via the identification $W(k/\kappa) = \langle F\rangle \cong \mathbf{Z}$, this element $g \in W(X_0,\overline{x})$ lifts the Frobenius $F \in W(k/\kappa)$. Then, $\rho(g)$ normalizes $\rho(\pi_1(X,\overline{x}))$, that is, $\rho(g)\rho(\pi_1(X,\overline{x}))\rho(g^{-1}) = \rho(\pi_1(X,\overline{x}))$, hence $\rho(g)$ normalizes the Zariski closure G_{geom} of $\rho(\pi_1(X,\overline{x}))$ by continiuity. This induces an action of $W(k/\kappa) = \langle F\rangle \circlearrowright G_{\mathrm{geom}}$ via conjugation.

Definition 5.22. We denote G_{arith} to be $G_{\mathrm{geom}} \rtimes W(k/\kappa) \cong G_{\mathrm{geom}} \rtimes \mathbf{Z}$, where the action of $\mathbf{Z}$ is via $1 \mapsto \rho(g)$ as above.

Note that the presentation of G_{arith} as a semidirect product depends on the choice of lifting, but the group G_{arith} does not. So we have a semidirect short exact sequence in the middle row below, such that the entire diagram commutes:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X,\overline{x}) \arrow[r] \arrow[d] & W(X_0,\overline{x}) \arrow[r] \arrow[d, "j"] & W(k/\kappa) \arrow[r] \arrow[d, "\wr"] & 1 \\
1 \arrow[r] & G_{\mathrm{geom}} \arrow[r] \arrow[rd, hook, "/" marking] & G_{\mathrm{arith}} \arrow[r, "\deg"] \arrow[d] & \mathbf{Z} \arrow[r] & 1 \\
 & & \mathrm{GL}(V) & &
\end{tikzcd} \tag{9}$$

Let us now assume $\mathscr{G}_0$ is semi-simple (for example, we may have started with a general lisse sheaf and then taken one of its composition factors). By the Tannakian duality between local systems and representations of π_1, this means that $\rho\colon \pi_1(X,\overline{x}) \rtimes \mathbf{Z} \cong W(X_0,\overline{x}) \to \mathrm{GL}(V)$ is semisimple. By Proposition 5.12, $\rho|_{\pi_1(X,\overline{x})}$ is semisimple. This implies that G_{geom} is reductive by Recollection 5.14(3), i.e., it has no normal unipotent subgroups.

We will show later that G_{geom} is semisimple, which means that all occurrences of the groups $Z(G_{\mathrm{geom}})$ and $(G_{\mathrm{geom}})_{\mathrm{ab}}$ below will be finite by Recollection 5.14(4). However, we first want the following result:

Lemma 5.23. *There is a positive integer N such that the semidirect sequence*

$$1 \longrightarrow G_{\mathrm{geom}} \longrightarrow \deg^{-1}(N\cdot\mathbf{Z}) \xrightarrow{\deg} N\cdot\mathbf{Z} \longrightarrow 1$$

from (9) *is direct, in other words,* $\deg^{-1}(N\cdot\mathbf{Z}) \cong G_{\mathrm{geom}} \times \mathbf{Z}$*.*

Once again, let $g \in W(X_0,\overline{x})$ be a chosen lift of $1 \in W(k/\kappa) \cong \mathbf{Z}$.

Proof. We set $G = G_{\mathrm{geom}}$ for brevity. The representation $\rho\colon Z(G) \to \mathrm{GL}(V)$ contains finitely many characters $\chi_1, \chi_2, \ldots, \chi_s$. Then, $\mathbf{Z}$ acts on $Z(G)$ by conjugation by $\rho(g) \in \mathrm{GL}(V)$, and thus must permute this list of characters. Replacing g by a power of g, we may assume this permutation is trivial, so that the action of $\mathbf{Z}$ on $Z(G)$ will be trivial.

Also, g acts on G_{adj}. The outer automorphism group of a semisimple group is the automorphism group of the Dynkin diagram, and thus likewise finite. So, replacing g by a power of g, we may assume that the action of g on G_{adj} is inner and, changing the choice of semidirect splitting, we may assume the action on G_{adj} is trivial.

After these reductions, the action on G is of the form $\left(\begin{smallmatrix} 1 & \eta \\ 0 & 1 \end{smallmatrix}\right)$ for some $\eta \in \mathrm{Hom}(G_{\mathrm{adj}}, Z(G))$. But there are no nontrivial maps from a connected semisimple group to a torus, so any map $G_{\mathrm{adj}} \to Z(G)$ is trivial on the connected component of the identity, and we see that $\mathrm{Hom}(G_{\mathrm{adj}}, Z(G))$ is likewise finite.

Passing to one more power of g, the action on G is now trivial and the product is direct. $\square$

We can now show the following:

Theorem 5.24 (Grothendieck)**.** *Let $\mathscr{G}_0$ be geometrically semisimple and lisse. Then,*

1. *G_{geom} and $G^{\circ}_{\mathrm{geom}}$ are semisimple;*
2. *Denoting $Z := Z(G_{\mathrm{arith}}(\overline{\mathbf{Q}}_\ell))$, the map $Z \overset{\psi}{\to} W(k/\kappa)$ has finite kernel and cokernel. More precisely, Z has a power of an element of degree 1.*

Corollary 5.25. *After base filed extension, $Z \overset{\psi}{\twoheadrightarrow} W(k/\kappa)$ is surjective.*

Proof of Theorem 5.24. For (1), choose N as in Lemma 5.23; replacing κ by its degree N extension, we can assume $N = 1$. So we have the following commutative diagram, where the second row is split by Lemma 5.23, π is the projection onto G_{geom}, and the dashed arrow is $\pi \circ \rho$:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X,\overline{x}) \arrow[r] \arrow[d, "\rho"] & W(X_0,\overline{x}) \arrow[r] \arrow[d, "\rho"] \arrow[dl, dashed] & \mathbf{Z} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & G_{\mathrm{geom}} \arrow[r, shift left] \arrow[d, "\alpha"] & G_{\mathrm{geom}} \times \mathbf{Z} \arrow[r] \arrow[l, shift left, "\pi"] & \mathbf{Z} \arrow[r] & 1 \\
 & (G_{\mathrm{geom}})_{\mathrm{ab}} & & &
\end{tikzcd}$$

Recall that $(G_{\mathrm{geom}})_{\mathrm{ab}}$ is the product of a finite group and a torus, and so applying Theorem 5.2 to $\alpha \circ \pi \circ \rho$, the image of $\pi_1(X, \overline{x})$ in $(G_{\mathrm{geom}})_{\mathrm{ab}}$ is finite. But it must also be Zariski dense by definition of G_{geom}. So $(G_{\mathrm{geom}})_{\mathrm{ab}}$ is finite and G_{geom} is semisimple. Passing to the identity component, we see that $G^{\circ}_{\mathrm{geom}}$ is also semisimple.

For (2), $\ker \psi \subset Z(G_{\mathrm{geom}}(\overline{\mathbf{Q}}_\ell))$ is finite since G_{geom} is semisimple. For the cokernel, we want to show that there exists $z \in Z(G_{\mathrm{arith}}(\overline{\mathbf{Q}}_\ell))$ such that $\deg(z) \neq 0$. As in (1), we can assume $N = 1$ in Lemma 5.23, and so there exists $\zeta \in G_{\mathrm{arith}}(\overline{\mathbf{Q}}_\ell)$ such that $\deg \zeta = 1$ and $\zeta \in C(G_{\mathrm{geom}}(\overline{\mathbf{Q}}_\ell))$, the centralized of $G_{\mathrm{geom}}(\overline{\mathbf{Q}}_\ell)$ in G_{arith}. Now for $g \in G_{\mathrm{geom}}(\overline{\mathbf{Q}}_\ell)$, define a cocycle

$$\begin{aligned} \phi\colon \mathbf{Z} &\longrightarrow G_{\mathrm{geom}}(\overline{\mathbf{Q}}_\ell) \\ n &\longmapsto g \cdot \zeta^n \cdot g^{-1} \zeta^{-n} \end{aligned}$$

This is a homomorphism since

$$\phi_g(n+m) = \phi_g(n) \zeta^n \phi_g(m) \zeta^{-n} = \phi_g(n)\phi_g(m),$$

where the first equality is by definition and the second is since $\zeta \in C(G_{\mathrm{geom}}(\overline{\mathbf{Q}}_\ell))$. Also, one can check that

$$g' \cdot \phi_g(n) g'^{-1} = \phi_g(n)$$

for all $g \in G_{\mathrm{geom}}(\overline{\mathbf{Q}}_\ell)$ and $g' \in G_{\mathrm{geom}}(\overline{\mathbf{Q}}_\ell)$. This implies $\operatorname{im} \phi_g \subset Z(G_{\mathrm{geom}}(\overline{\mathbf{Q}}_\ell))$, which is finite since G_{geom} is semisimple, and so there there exists $n > 0$ such that $\phi_g(n) = 1$ for all $g \in G_{\mathrm{geom}}(\overline{\mathbf{Q}}_\ell)$. Thus, $\zeta^n \in C(G_{\mathrm{geom}}(\overline{\mathbf{Q}}_\ell))$, and $\zeta^n \in Z(G_{\mathrm{arith}}(\overline{\mathbf{Q}}_\ell))$ since $G_{\mathrm{arith}}(\overline{\mathbf{Q}}_\ell)$ is generated by ζ and $G_{\mathrm{geom}}(\overline{\mathbf{Q}}_\ell)$. Finally, setting $z := \zeta^n$ gives us our desired element. ☐

5.3 Applications

Lemma 5.26. *Let $\mathscr{G}_0$ be semisimple and lisse, and let $z \in Z(G_{arith}(\overline{\mathbf{Q}}_\ell))$ with $\deg(z) = n \neq 0$, as in the proof of Theorem 5.24(2). Suppose z acts on $V = \mathscr{G}_{0\overline{x}}$ with eigenvalues $\alpha_1, \ldots, \alpha_r$, where*

$$|\tau(\alpha_i)^2| = q^{n\beta_i}.$$

Then, $\beta_1, \ldots, \beta_r$ are determinant weights of $\mathscr{G}_0$.

Proof. $z \in Z(G_{\mathrm{arith}}(\overline{\mathbf{Q}}_\ell))$ implies that z is a homomorphism of G-modules. Then, by Schur's lemma, $\rho(z)$ acts by eigenvalues on irreducible summands, and so reducing to the case when $\mathscr{G}_0$ is irreducible, we have that z acts by some eigenvalue α. So z acts on $\bigwedge^{\dim V} V$ by $\alpha^{\dim V}$. Looking at the definition of determinant weights, we have $q^{n\beta_i} = |\tau(\alpha)^2|$. ☐

We also conclude that determinant weights behave in the obvious way for pullback, direct sum, and tensor product, by considering how the eigenvalues of z change for these operations:

Theorem 5.27. *Let $f_0 \colon X_0' \to X_0$ be a dominant morphism between normal, geometrically connected schemes, and let $\mathscr{F}_0$ and $\mathscr{G}_0$ be lisse sheaves on X_0. Then,*

1. *$\mathscr{G}_0$, $f_0^*\mathscr{G}_0$ have the same determinant weights;*
2. *If $\mathscr{F}_0$, $\mathscr{G}_0$ each have single determinant weights α, β, then $\mathscr{F}_0 \otimes \mathscr{G}_0$ has determinant weight $\alpha + \beta$;*
3. *For all weights γ, denoting $r(\gamma)$ to be the sum of ranks of irreducible constituents with weight γ, the determinant weights of $\bigwedge^r \mathscr{F}_0$ are*

$$\sum_{\gamma \in \mathbf{R}} n(\gamma) \cdot \gamma, \quad \textit{where} \quad \sum n(\gamma) = r,\ 0 \le n(\gamma) \le r(\gamma),\ \textit{and } n(\gamma) \in \mathbf{Z}.$$

Proof of (2) and (3). Replace $\mathscr{F}_0, \mathscr{G}_0$ with their semisimplifications, and use Lemma 5.26. ☐

6 May 30—Real Sheaves (Matt Stevenson)

Notation 6.1. Unless otherwise stated, $\kappa = \mathbf{F}_q$, $k = \overline{\kappa}$; X_0 is an algebraic (i.e., finite type) scheme over κ; $\mathscr{G}_0$ is a (Weil) sheaf on X_0; $\tau \colon \overline{\mathbf{Q}}_\ell \xrightarrow{\sim} \mathbf{C}$ is a fixed isomorphism; and if $x \in |X_0|$ is a closed point, then $\overline{x}$ is a geometric point over x, $d(x) = [\kappa(x) : \kappa]$, and $N(x) = \#\kappa(x) = q^{d(x)}$.

We recall something Brandon talked about a while ago (see Definition 3.12). Recall that $\mathscr{G}_0$ is *τ-real* if for all $x \in |X_0|$, the characteristic polynomial of geometric Frobenius $F_x \colon \mathscr{G}_{0\overline{x}} \to \mathscr{G}_{0\overline{x}}$ has real coefficients, i.e., if $\tau \det(1 - F_x t,\ \mathscr{G}_{0\overline{x}}) \in \mathbf{R}[t]$.

Goal 6.2. We want to show τ-real sheaves are in fact τ-mixed, and that the determinant weights of a τ-real sheaf are the τ-weights.

We will do this for $X_0, \mathscr{G}_0$ satisfying some additional hypotheses.

6.1 Curve case

Let X_0 be a smooth geometrically irreducible curve over κ, and let $\mathscr{G}_0$ be a smooth sheaf on X_0. Then, the Weil group $W(X_0, \overline{x})$ acts on $V = \mathscr{G}_{0\overline{x}}$, and $\pi \coloneqq \pi_1(X, \overline{x}) \subset W(X_0, \overline{x})$.

Remark 6.3. $H^2_c(X, \mathscr{G}) = V_\pi(-1)$ since they are Poincaré dual to $H^0(X, \mathscr{G}) = V^\pi$, and so if α is an eigenvalue of Frobenius $F \circlearrowright H^2_c(X, \mathscr{G})$, then αq^{-1} is an eigenvalue of $F \circlearrowright V_\pi$. Thus,

$$\frac{\log\Big(\big|\tau\big(\alpha q^{-1}\big)\big|^2\Big)}{\log(q)}$$

is a determinant weight of $\mathscr{F}_0 = \widetilde{V_\pi}$ on $\operatorname{Spec} \kappa$. By Theorem 5.27, this is a determinant weight of $\mathscr{F}_0'$, the pullback of $\mathscr{F}_0$ on X_0, hence is also a determinant weight of $\mathscr{G}_0$.

This is saying that if we have an idea for what eigenvalues of the Frobenius look like, then we also have an idea for what determinant weights look like.

Remark 6.4. If $\mathscr{G}_0$ is τ-real, then the logarithmic derivative of $\tau \det(1 - F_x t,\ \mathscr{G}_{0\overline{x}}^{\otimes k})^{-1}$ is

$$f(t) = \sum_{n=1}^{\infty} \tau\Big(\operatorname{Tr}(A^n)^k\Big) \cdot t^{n-1},$$

where A is the matrix of the Frobenius on $\mathscr{G}_{0\overline{x}}$. Since $\mathscr{G}_0$ is τ-real, all of these $\tau\Big(\operatorname{Tr}(A^n)^k\Big)$ are real, and so the power series

$$\tau\det(1-F_xt,\ \mathscr{G}_{0\overline{x}}^{\otimes k})^{-1} = e^{\int f(t)\,dt}$$

has real coefficients, so $\mathscr{G}_0^{\otimes k}$ is τ-real for all $k \geq 1$. In particular, the coefficients of $\tau\det\big(1-F_xt,\ \mathscr{G}_{0\overline{x}}^{\otimes 2k}\big)^{-1}$ are in $\mathbf{R}_{\geq 0}$.

We can now talk about our main theorem in the curve case.

Theorem 6.5 (Rankin–Selberg Method)**.** *Let X_0 be a smooth, geometrically irreducible curve over κ, and $\mathscr{G}_0$ be a smooth sheaf over X_0 that is τ-real.. Then, all irreducible constituents of $\mathscr{G}_0$ are τ-pure, and their τ-weights coincide with their determinant weights.*

Remark 6.6. Deligne does not refer to this result as the Rankin–Selberg method. Katz mentions this result as Rankin's method in [Kat01, Introduction]. Lauman may also refer to this as Rankin's method.

We first need an a priori estimate for eigenvalues of Frobenius.

Lemma 6.7. *Let $X_0, \mathscr{G}_0$ be as in Theorem 6.5, let α be an eigenvalue of $F_x \circlearrowright \mathscr{G}_{0\overline{x}}$, and let β be the largest determinant weight of $\mathscr{G}_0$. Then,*

$$|\tau(\alpha)|^2 \leq N(x)^{\beta}. \tag{10}$$

Proof. Assume $\mathscr{G}_0 \neq 0$, and X_0 is affine (so that $H_c^0(X,\mathscr{G}) = 0$). Then, for all $k \geq 1$, $2k\beta$ is the largest determinant weight of $\mathscr{G}_0^{\otimes 2k}$ by Theorem 5.27. If t_0 is a zero of $\tau\det(1-F_xt,\ H_c^2(X,\mathscr{G}^{\otimes 2k}))$, then

$$\frac{\log\left(|t_0^{-1}q^{-1}|^2\right)}{\log(q)}$$

is a determinant weight of $\mathscr{G}_0^{\otimes 2k}$, hence $\leq 2k\beta$. Equivalently, $|t_0| \geq q^{-(2k\beta+2)/2}$.

Now we write down the Grothendieck trace formula as a quotient of two characteristic polynomials; this will let us determine when the infinite product may or may not converge.

For $k \geq 1$, the Grothendieck trace formula for $\mathscr{G}^{\otimes 2k}$ says

$$\prod_{x\in|X_0|} \tau\det(1-F_xt^{d(x)},\ \mathscr{G}_{0\overline{x}}^{\otimes 2k})^{-1} = \frac{\tau\det(1-Ft,\ H_c^1(X,\mathscr{G}^{\otimes 2k}))}{\tau\det(1-Ft,\ H_c^2(X,\mathscr{G}^{\otimes 2k}))}.$$

The denominator on the right-hand side has no zeros in the disk $|t| < q^{-(2k\beta+2)/2}$, and so the infinite product on the left-hand side converges there. Thus, each of the "local L-factors" also converge there (since if you have a complex power series which is a product of power series with leading term one, with non-negative real coefficients (using that our power $2k$ is even), then the radius of convergence of the entire product is at most the radius of convergence of each factor). In particular, these polynomials $\tau\det(1-F_xt^{d(x)}, \mathscr{G}_{0\overline{x}}^{\otimes 2k})$ are zero-free in the region $|t| < q^{-(2k\beta+2)/2}$.

Now given an eigenvalue α of $F_x \circlearrowright \mathscr{G}_{0\overline{x}}$, its $(2k)$th power α^{2k} is an eigenvalue of $F_x \circlearrowright \mathscr{G}_{0\overline{x}}^{\otimes 2k}$. So, this formula tells us that

$$|\tau(\alpha^{-2k/d(x)})| \geq q^{-(2k\beta+2)/2}$$

Equivalently,

$$|\tau(\alpha)|^2 \leq q^{d(x)(2k\beta+2)/2k} = N(x)^{\beta+\frac{1}{k}}.$$

Finally, send $k \to +\infty$. □

We are now ready to prove the Theorem.

Proof of Theorem 6.5. For $\beta \in \mathbf{R}$, we will denote $\mathscr{G}_0(\beta)$ to be the direct sum of irreducible constituents of $\mathscr{G}_0$ with determinant weight β, and denote $r(\beta) = \operatorname{rank}\mathscr{G}_0(\beta)$. Now for a fixed closed point $x \in |X_0|$, the eigenvalues of $\mathscr{G}_0$ will appear as eigenvalues of some $\mathscr{G}_0(\beta)$, so let

$$\alpha_1^{\beta}, \ldots, \alpha_{r(\beta)}^{\beta}$$

be the eigenvalues of $F_x \circlearrowright \mathscr{G}_0(\beta)_{\overline{x}}$. We want to show that $|\tau(\alpha_i^{\beta})|^2 = N(x)^{\beta}$ for all $i = 1, \ldots, r(\beta)$. We would want to just apply Lemma 6.7 to both $\mathscr{G}_0$ and its dual $\mathscr{G}_0^{\vee}$ to get inequalities in both directions, but it turns out we need to modify $\mathscr{G}_0$ a bit first for this to actually be possible.

Observation 6.8. The determinant weight β of $\mathscr{G}_0(\beta)$ can be written in terms of the α_j^β, by definition of determinant weights:

$$\beta = \frac{1}{r(\beta)} \sum_{j=1}^{r(\beta)} \frac{\log\left(\left|\tau\left(\alpha_j^\beta\right)\right|^2\right)}{\log(N(x))}.$$

Equivalently,

$$\left|\tau\left(\prod_{j=1}^{r(\beta)} \alpha_j^\beta\right)\right|^2 = N(x)^{\beta\cdot r(\beta)}.$$

Now, letting $N = \sum_{\gamma>\beta} r(\gamma)$ (which is a finite sum since there are only finitely many determinant weights in $\mathscr{G}_0$), we know by Theorem 5.27(3) what the determinant weights of $\bigwedge^{N+1}\mathscr{G}_0$ look like:

- $\mathscr{G}_0 \wedge \left(\bigwedge^N \mathscr{G}_0\right) = \bigwedge^{N+1}\mathscr{G}_0$ has largest determinant weight

  $$\beta + \sum_{\gamma>\beta} r(\gamma)\cdot\gamma.$$

  We can get many more determinant weights by lowering $r(\gamma)$.
- An eigenvalue of F_x on $\bigwedge^{N+1}\mathscr{G}_0$ is

  $$\alpha_i^\beta \prod_{\gamma>\beta}\prod_{j=1}^{r(\gamma)} \alpha_j^\gamma.$$

Now (10) tells us that

$$\left|\tau\left(\alpha_i^\beta \prod_{\gamma>\beta}\prod_{j=1}^{r(\gamma)} \alpha_j^\gamma\right)\right|^2 \le N(x)^{\beta+\sum_{\gamma>\beta} r(\gamma)\cdot\gamma}.$$

Observation 6.8 implies that, by pulling the multiplication out,

$$\left|\tau\left(\prod_{\gamma>\beta}\prod_{j=1}^{r(\gamma)} \alpha_j^\gamma\right)\right|^2 = \prod_{\gamma>\beta}\left|\tau\left(\prod_{j=1}^{r(\gamma)} \alpha_j^\gamma\right)\right|^2 = \prod_{\gamma>\beta} N(x)^{r(\gamma)\cdot\gamma}.$$

Thus, $|\tau(\alpha_i^\beta)|^2 \le N(x)^\beta$.

Now we can get the opposite inequality by replacing $\mathscr{G}$ with its dual $\mathscr{G}^\vee$. ☐

6.2 General Case

Theorem 6.9. *Let X_0 be an algebraic scheme, $\mathscr{G}_0$ be a τ-real sheaf. Then,*

1. *$\mathscr{G}_0$ is τ-mixed;*
2. *[Purity] Let X_0 be irreducible and normal, and let $\mathscr{G}_0$ be smooth. Then, the irreducible constituents of $\mathscr{G}_0$ are τ-pure of the appropriate weight (that is, equal to their determinant weight).*

Proof. (2) follows from (1) and Theorem 3.10(3). This isn't completely formal since you need to understand the proof of (1) to get (2).

For (1), first we note the following:

Observation 6.10. Let $j_0\colon U_0 \hookrightarrow X_0$ be open, and let $i_0\colon S_0 \hookrightarrow X_0$ be the closed complement of U_0. Then, we have a short exact sequence

$$0 \longrightarrow j_{0!}j_0^*\mathscr{G}_0 \longrightarrow \mathscr{G}_0 \longrightarrow i_{0*}i_0^*\mathscr{G}_0 \longrightarrow 0,$$

and so it suffices to show $j_0^*\mathscr{G}_0$ and $i_0^*\mathscr{G}_0$ are τ-mixed. Since we have already shown the dimension 1 case in Theorem 6.5, we know $i_0^*\mathscr{G}_0$ is τ-mixed already by noetherian induction.

Observation 6.11. We claim we can freely pass to finite field extensions of our base field. Let κ'/κ be a finite extension, and write $X_0' = X_0 \otimes_\kappa \kappa'$. If $\mathscr{G}_0'$, the pullback of $\mathscr{G}_0$ to X_0', is τ-mixed, then its direct image in X_0 will be τ-mixed, and $\mathscr{G}_0$ will be a subsheaf of this direct image, hence will also be τ-mixed (this uses facts from Permanence Properties 3.4).

These two facts (and reductions similar to what we have been doing so far; see [KW01, §I.3]) allow us to reduce to the following case: X_0 is smooth, irreducible, and affine; $\mathscr{G}_0$ is smooth; $\dim X_0 > 1$; and all irreducible constituents of $\mathscr{G}_0$ are geometrically irreducible.

Now we can embed X_0 into some projective space $\mathbf{P}_0^N$ over κ, and let $\mathscr{F}_0$ be an irreducible constituent of $\mathscr{G}_0$. We want to show that we can find an open subset $U_0 \subset X$ such that $\mathscr{F}|_{U_0}$ is τ-pure.

The idea is as follows. We use the geometric irreducibility assumption to be able to base change to the algebraic closure k, and use (a version of) Bertini's theorem. This allows us to find an open set $U \subset X$ over k where (1) holds, and then we can hope to descend this U to an open set defined over κ.

Consider linear subspaces $L \subset \mathbf{P}^N$ (over $k = \overline{\kappa}$) of codimension $\dim X_0 - 1$, so that $X \cap L$ is (generically) a curve. These linear subspaces will be the k-points of some Grassmannian G. We can consider those particular L's that satisfy some nice properties: $L \cap X =: C$ is a nonempty, smooth, irreducible curve; and $\mathscr{F}|_C$ is irreducible. The Bertini theorem for Weil sheaves (see [KW01, Thm. B.1]) says these L's are the k-points of a nonempty open subset $\Omega \subset G$. And for such an $L \in \Omega$, there exists a finite field extension κ'/κ such that C is defined over κ', i.e., there exists $C_0 \subset X_0 \otimes_\kappa \kappa'$ a closed curve such that $C = C_0 \otimes_{\kappa'} \kappa$.

Let $\mathscr{F}_0'$ be the pullback of $\mathscr{F}_0$ to C_0, and let $\mathscr{G}_0'$ be the pullback of $\mathscr{G}_0$ to C_0. Then, $\mathscr{G}_0'$ is still τ-real, $\mathscr{F}_0'$ is still irreducible, and $\mathscr{F}_0'$ is still an irreducible constituent of $\mathscr{G}_0'$. Now the Rankin–Selberg method (Theorem 6.5) says that $\mathscr{F}_0$ is τ-pure of the appropriate weight.

To finish our proof of the theorem, we allow $L \in \Omega$ to vary. There exists a nonempty open $U \subset X$ such that every k-point of U is contained in at least one of these $L \in \Omega$. We can assume that U is defined over κ since it is defined over some finite extension of κ, and so you can replace U by the intersection of its Galois conjugates to get an open subset defined over κ. We then get a subset $U_0 \subset X_0$ such that its base change is an open subset U as above (which might be smaller than the original U). □

Remark 6.12. It may be possible to avoid passing to the algebraic closure k by using Poonen's Bertini theorems over finite fields.

7 June 1—Fourier Transforms (Charlotte Chan)

We will first be very concrete by reviewing Fourier transforms of functions defined over finite fields.

The classical Fourier transform is defined over $\mathbf{R}$, using integrals. The analogue for finite fields $\mathbf{F}_q$ is instead defined by sums.

Notation 7.1. We will fix for today a non-trivial (i.e., not always equal to 1) additive character

$$\psi\colon \mathbf{F}_q \longrightarrow \mathbf{C}^*.$$

Note that this induces characters on every finite extension $\mathbf{F}_{q^n}$ by

$$\psi\colon \mathbf{F}_{q^n} \xrightarrow{\mathrm{Tr}} \mathbf{F}_q \xrightarrow{\psi} \mathbf{C}^*,$$

which we will also denote by ψ.

We will fix an isomorphism $\tau\colon \overline{\mathbf{Q}}_\ell \xrightarrow{\sim} \mathbf{C}$, and for today will suppress this notation so we don't get bogged down in notation. In addition, as always, $\kappa = \mathbf{F}_q$, $\kappa_n = \mathbf{F}_{q^n}$, and $k = \overline{\mathbf{F}_q}$.

We start by recalling the definition of the Fourier transform for functions defined over finite fields.

Definition 7.2 (Fourier transform over finite fields)**.** Let $f\colon \mathbf{F}_{q^n} \to \mathbf{C}$. Then, the *Fourier transform* of f is defined as the function

$$\begin{aligned} T_\psi f\colon \mathbf{F}_{q^n} &\longrightarrow \mathbf{C} \\ x &\longmapsto \sum_{y \in \mathbf{F}_{q^n}} f(y)\psi(-xy). \end{aligned} \tag{11}$$

As in the real case, we have the following formulas:

Theorem 7.3 (Plancherel Formula)**.** $\|T_\psi f\|_n^2 = q^n \|f\|_n^2$, *where we recall*

$$\|f\|_n^2 = (f,f)_n = \sum_{x \in \mathbf{F}_{q^n}} f(x)\overline{f(x)}.$$

Theorem 7.4 (Fourier Inversion)**.** $T_{\psi^{-1}} T_\psi f = q^n f$.

Our goal is to use the "sheaf-to-functions" correspondence to develop an analogue of the Fourier transform for complexes of sheaves. More precisely:

Goal 7.5. Letting $K_0 \in \mathsf{D}_c^b(\mathbf{A}_0^1, \overline{\mathbf{Q}}_\ell)$, we want to construct another complex $T_\psi K_0$ whose corresponding function satisfies Theorems 7.3 and 7.4.

Recall that a subscript 0 denotes that the object in question is defined over $\kappa = \mathbf{F}_q$.

We start by recalling the sheaf-to-functions correspondence.

Recall 7.6 (Sheaf-to-functions correspondence)**.** To $K_0 \in \mathsf{D}_c^b(\mathbf{A}_0^1, \overline{\mathbf{Q}}_\ell)$, we associate the function

$$\begin{aligned} f^{K_0} \colon \mathbf{F}_{q^n} &\longrightarrow \mathbf{C} \\ x &\longmapsto \sum_i (-1)^i \operatorname{Tr}\big(F_x,\, \mathscr{H}^i(K_0)\big)\,. \end{aligned}$$

To get analogues of Theorems 7.3 and 7.4, what we will do is develop proofs side by side with the proofs of the results for functions to get an idea for how to prove the sheaf-theoretic versions.

Recollections 7.7. Before we begin, we recall the following facts about constructible sheaves:

1. Grothendieck trace formula: For a constructible sheaf $\mathscr{F}/X_0$, we have

$$\sum_{x \in X_0(\mathbf{F}_{q^n})} \operatorname{Tr}(F_x,\, \mathscr{F}) = \sum_i (-1)^i \operatorname{Tr}(F_{q^n},\, H_c^i(X, \mathscr{F})).$$

2. Base change: Consider the following cartesian square of finite type schemes over $\mathbf{F}_q$:

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \arrow[dl, dashed] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Then, "going along the dashed arrow in either direction gives the same result," that is, there is a natural isomorphism of functors

$$\mathbf{R}f'_! \circ g'^* \cong g^* \circ \mathbf{R}f_!.$$

3. Special case of base change: If $Y' = \{y\}$ is a point in Y, then the cartesian square in (2):

$$\begin{tikzcd} X_y \arrow[r, "i"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ \{y\} \arrow[r, "i_y"] & Y \end{tikzcd}$$

gives, for a complex K_0 on X,

$$\mathbf{R}f'_! i^* K_0 \cong i_y^* \mathbf{R}f_! K_0 = \text{the stalk of } \mathbf{R}f_! K_0 \text{ at } y.$$

But the left-hand side is also equal to $\mathbf{R}\Gamma_c\big(K_0|_{X_y}\big)$, and so this gives the isomorphism

$$\text{the stalk of } R^i f_! K_0 \text{ at } y \cong H_c^i(X_y, K_0).$$

4. Combining (1) and (3): For a morphism $f\colon X \to Y$ and a complex K_0 on X,
$$f^{\mathbf{R}f_! K_0}(y) = \sum_{x \in X_y(\mathbf{F}_{q^n})} f^{K_0}(x),$$
where y is a $\mathbf{F}_{q^n}$-point, by using the sheaf-to-functions correspondence.
5. We also translate (2) to the language of functions. If $y' \in Y'$, then
$$\sum_{x' \in X'_{y'}} f^{K_0}(g'(x')) = \sum_{x \in X_{g(y')}} f^{K_0}(x).$$
6. Projection formula: If $f\colon X \to Y$ is a morphism, and M, N are defined on X, Y, respectively, then
$$\mathbf{R}f_!(f^*N \otimes M) \cong N \otimes \mathbf{R}f_! M,$$
and so pullback and pushforward have some compatibility. In terms of functions,
$$\sum_{x \in X_y} f^N(f(x)) \cdot f^M(x) = \sum_{x \in X_y} f^N(y) \cdot f^M(x) = f^N(y) \sum_{x \in X_y} f^M(x).$$
Note this is an example of where the result for functions is trivial, but the proof on the sheaf side is more complicated.

These facts will help us prove analogues of Theorems 7.3 and 7.4 for complexes; the sums on the function side will help us understand where the proofs come from.

We first need to define a sheaf associated to ψ. To do so, we use Artin–Schreier sheaves. Consider the morphism
$$\begin{aligned} \wp\colon \mathbf{A}^1_0 &\longrightarrow \mathbf{A}^1_0 \\ x &\longmapsto x^q - x. \end{aligned}$$
This is a finite Galois étale covering, with Galois group $\mathbf{F}_q$, and so we get a surjection
$$\pi_1(\mathbf{A}^1, \overline{x}) \twoheadrightarrow \mathbf{F}_q.$$
So, attached to ψ, we get a rank one étale sheaf $\mathscr{L}_0(\psi)$ called the *Artin–Schreier sheaf* on $\mathbf{A}^1_0$, since we have a character
$$\pi_1(\mathbf{A}^1, \overline{x}) \longrightarrow \mathbf{F}_q \xrightarrow{\psi} \mathbf{C}^*$$
of the fundamental group.

Remark 7.8. If we base change $\mathscr{L}_0(\psi)$ to $\mathbf{F}_{q^n}$, we get $\mathscr{L}_0(\operatorname{Tr} \circ \psi)$, that is
$$\mathscr{L}_0(\psi) \otimes \mathbf{F}_{q^n} = \mathscr{L}_0(\operatorname{Tr} \circ \psi).$$
See [KW01, pp. 39–40].

Now that we have defined a sheaf, the logical thing to ask is what function this sheaf corresponds to.

Lemma 7.9. *$f^{\mathscr{L}_0(\psi)}(x) = \psi(-x)$. In particular, the weight $w(\mathscr{L}_0(\psi)) = 0$.*

Note that $\mathscr{L}_0(\psi)$ is τ-pure, so $w(\mathscr{L}_0(\psi))$ is the unique weight of $\mathscr{L}_0(\psi)$.

Proof. Let $x \in \mathbf{F}_{q^n} = \mathbf{A}^1_0 \otimes \mathbf{F}_{q^n}$. Then, consider the arithmetic Frobenius
$$\begin{aligned} \sigma\colon \overline{\mathbf{F}}_q &\longrightarrow \overline{\mathbf{F}}_q \\ \alpha &\longmapsto \alpha^{q^n} \end{aligned}$$
Then, if α satisfies $\alpha^{q^n} - \alpha = x$, then also $\sigma(\alpha)^{q^n} - \alpha = x$. On the other hand, $\sigma(\alpha) - \alpha = x$, and so $\sigma(\alpha) = x + \alpha$, i.e., σ corresponds to addition by x. Thus, the geometric Frobenius F_x, which is the inverse of the arithmetic Frobenius, corresponds to subtraction by x, that is, it translates $\alpha \mapsto \alpha - x$. In particular, F_x acts on $\mathscr{L}_0(\psi)$ by $\psi(-x)$. $\square$

We have a way of getting sums and products of functions via operations on the sheaf side from Recollections 7.7, and so we put them together to define an analogue of the Fourier transform in the sheaf case. Note that our naïve guess is not completely right and we need to introduce a shift of our complex.

Notation 7.10. Round brackets (1) denote Tate twists, that is, $K(1) \coloneqq K \otimes \overline{\mathbf{Q}}_\ell(1)$, and square brackets [1] denote the degree shift to the left for complexes, i.e., $K[1]^i \coloneqq K^{i+1}$.

Definition 7.11. Consider the following diagram

$$\begin{tikzcd}
 & \mathbf{A}^1_0 \times \mathbf{A}^1_0 \arrow[rr, "m"] \arrow[dl, "\pi^1"'] \arrow[dr, "\pi^2"] & & \mathbf{A}^1_0 \;\; \mathscr{L}_0(\psi) \\
\mathbf{A}^1_0 & & \mathbf{A}^1_0 & \\
 & & K_0 &
\end{tikzcd}$$

where m denotes the multiplication map $(x, y) \mapsto xy$. Consider the sheaf $\mathscr{L}_0(\psi)$ defined on the codomain of m, and K_0 defined on the codomain of π^2. Then, we define the *functor Fourier transform*

$$T_\psi \colon \mathsf{D}^b_c(\mathbf{A}^1_0, \overline{\mathbf{Q}}_\ell) \longrightarrow \mathsf{D}^b_c(\mathbf{A}^1_0, \overline{\mathbf{Q}}_\ell)$$

by

$$T_\psi K_0 = \mathbf{R}\pi^1_! \big(\pi^{2*} K_0 \otimes m^* \mathscr{L}_0(\psi)\big)\,[1].$$

Later we will see why this shift [1] is needed: it fixes the perversity.

This is a direct analogue of the Definition 7.2, except for the shift [1]. If we wanted to make Definition 7.2 match our new definition, we would need to introduce a minus sign at the front of (11):

Lemma 7.12. *$f^{T_\psi K_0}(x) = -\sum_{y \in \mathbf{F}_{q^n}} f^{K_0}(y)\psi(-xy)$ for $x \in \mathbf{F}_{q^n}$.*

Proof. By definition,

$$f^{T_\psi K_0}(x) = \sum_i (-1)^i \operatorname{Tr}\big(F_x,\, \mathscr{H}^i\big(\mathbf{R}\pi^1_!(\pi^{2*} K_0 \otimes m^* \mathscr{L}_0(\psi))[1]\big)\big)$$

By pulling out the shift [1] and using Recollection 7.7(4),

$$\begin{aligned}
&= -\sum_i (-1)^i \sum_{y \in \mathbf{F}_{q^n}} \operatorname{Tr}\big(F_{(x,y)},\, \mathscr{H}^i(\pi^{2*} K_0 \otimes m^* \mathscr{L}_0(\psi))\big) \\
&= -\sum_{y \in \mathbf{F}_{q^n}} \sum_i (-1)^i \operatorname{Tr}\big(F_y,\, \mathscr{H}^i(K_0)\big) \cdot \psi(-xy) \\
&= -\sum_{y \in \mathbf{F}_{q^n}} f^{K_0}(y) \cdot \psi(-xy).
\end{aligned}$$

□

The following is *very* important, even if the proof is short.

Theorem 7.13. *Let $a \in \overline{\mathbf{F}}_q$ be a geometric point of $\mathbf{A}^1 = \mathbf{A}^1_0 \otimes \overline{\mathbf{F}}_q$. Then,*

$$\big(T_\psi(K_0)\big)_a = \mathbf{R}\Gamma_c(K \otimes \mathscr{L}(\psi_a))\,[1],$$

where $\psi_a \colon \mathbf{F}_{q^n} \to \mathbf{C}$ maps $x \mapsto \psi(ax)$, and $\mathbf{F}_{q^n}$ is chosen such that it contains a. The (complexes of) sheaves on the right are the pullbacks of those with subscripts 0 to $\mathbf{A}^1 = \mathbf{A}^1_0 \otimes \overline{\mathbf{F}}_q$.

Proof. Using Recollection 7.7(3) (base change) above,

$$\big(T_\psi(K_0)\big)_a = \mathbf{R}\Gamma_c\big(\big(\pi^{2*} K \otimes m^* \mathscr{L}(\psi)\big)\big|_{\{a\} \times \mathbf{A}^1}\big)[1] = \mathbf{R}\Gamma_c(K \otimes \mathscr{L}(\psi_a))\,[1].$$

□

Remark 7.14. This Theorem is very important for the purposes of the seminar: recall that what we want to study is Frobenius actions on cohomology $H^i_c(\mathbf{A}^1, K)$. Now, we've taken these cohomology spaces and stuck them into a nice family (a sheaf), so that we can view $T_\psi(K_0)$ as a "deformation" of $H^\bullet_c(\mathbf{A}^1, K)$. Taking stalks will eventually give the information we wanted:

$$\mathscr{H}^i((T_\psi(K_0))_0) = H^i_c(\mathbf{A}^1, K),$$

since $\mathbf{R}\Gamma_c(-)$ gives the complex used to compute cohomology. This will be used crucially in the next talk.

We are now ready to show the analogue of Theorem 7.3:

Theorem 7.15 (Plancherel Formula)**.** $\|f^{T_\psi(K_0)}\|_n = q^{n/2}\|f^{K_0}\|_n$.

Remark 7.16. We have that, assuming $\mathscr{H}^i(K_0)$ are all τ-mixed,

$$w(K_0) = \max\bigl\{w\bigl(\mathscr{H}^i(K_0)\bigr) - i\bigr\}.$$

Thus, the Plancherel formula above can be interpreted in terms of weights, using Theorem 4.11: $w(T_\psi(K_0)) = w(K_0) + 1$.

Proof of Theorem 7.15. By definition and Lemma 7.12,

$$\|f^{T_\psi(K_0)}\|_n^2 = \sum_{x\in\mathbf{F}_{q^n}} f^{T_\psi(K_0)}(x)\overline{f^{T_\psi(K_0)}(x)} = \sum_{x,y,z\in\mathbf{F}_{q^n}} f^{K_0}(y)\overline{f^{K_0}(z)}\underbrace{\psi(-xy)\psi(xz)}_{\psi(x(z-y))}.$$

Now note that

$$\sum_x \psi(x(z-y)) = \begin{cases} 0 & z \neq y \\ q^n & z = y \end{cases} \tag{12}$$

and so

$$\|f^{T_\psi(K_0)}\|_n^2 = \sum_{\substack{x,y,z\in\mathbf{F}_{q^n}\\ z=y}} f^{K_0}(y)\overline{f^{K_0}(z)}\psi(0) = q^n(f,f)_n = q^n\|f^{K_0}\|_n^2. \qquad \square$$

We can rephrase the calculation (12) in terms of the Fourier transform of the constant function 1:

$$T_\psi(1) = \sum_{x\in\mathbf{F}_{q^n}} \psi(xy) = \begin{cases} 0 & y \neq 0 \\ q^n & y = 0 \end{cases}$$

This tells us what we shoud expect the Fourier transform of the constant sheaf $\overline{\mathbf{Q}}_\ell$ should be:

Lemma 7.17. *Let $\delta_0 \coloneqq i_{0*}\overline{\mathbf{Q}}_\ell$ be the skyscraper sheaf, where $i_0\colon \{0\} \hookrightarrow \mathbf{A}^1$. Then,*

$$T_\psi(\overline{\mathbf{Q}}_\ell[1]) = \delta_0(-1).$$

Proof of Lemma 7.17. We use the following:

Fact 7.18 ([KW01, p. 42])**.** Using the Leray spectral sequence for the Artin–Schreier cover $\wp\colon x \mapsto x^q - x$, together with the fact that

$$\wp_*\overline{\mathbf{Q}}_\ell \cong \bigoplus_{x\in\mathbf{F}_q} \mathscr{L}(\psi_x),$$

one can show

$$H^i_c(\mathbf{A}^1, \mathscr{L}(\psi_x)) = \begin{cases} \overline{\mathbf{Q}}_\ell(-1) & x = 0,\ i = 2 \\ 0 & \text{else} \end{cases}$$

Now use base change (Recollection 7.7(3)):

$$\mathbf{R}\pi^1_!(m^*\mathscr{L}_0(\psi))_x[1] = \mathbf{R}\Gamma_c(\mathscr{L}(\psi_x))[1] = \delta_0(-1)[-1]. \qquad \square$$

Remark 7.19 (Perversity). Fix middle perversity. The sheaf $\overline{\mathbf{Q}}_\ell[1]$ is perverse and so is $\delta_0(-1)$, since we are pushing forward from a point. So the [1] is important to preserve perversity.

We will now prove the sheaf analogue of Fourier Inversion (Theorem 7.4), using the sum formulation as guidance.

Theorem 7.20 (Fourier Inversion)**.** $T_{\psi^{-1}} T_\psi K_0 = K_0(-1)$

The Tate twist (-1) multiplies the geometric Frobenius F_{q^n} action by q^n, and this corresponds to the q^n factor in the function case.

We will use base change and the projection formula repeatedly in the proof, so we will only refer to them by name, not by their references, i.e., Recollections 7.7(3) and (6).

Proof. We will develop the proof by writing down the proofs for sheaves and functions side-by-side; for convenience, on the function side we will write $f = f^{K_0}$. First, by definition

$$T_{\psi^{-1}} T_\psi K_0 \qquad\qquad \left(T_{\psi^{-1}} T_\psi f\right)(x)$$

$$= \mathbf{R}\pi^1_!\left(\pi^{2*}\left(\mathbf{R}\pi^1_!(\pi^{2*}K_0 \otimes m^*\mathscr{L}_0(\psi))\right) \otimes m^*\mathscr{L}_0(\psi^{-1})\right)[2] \qquad = \sum_y \left(\sum_z f(z)\psi(-yz)\right)\psi(xy)$$

We want to remove the parentheses on the function side. To perform the corresponding operation on sheaves, we consider the following diagram:

$$\begin{tikzcd}
 & & \mathbf{A}^1_0 \times \mathbf{A}^1_0 \times \mathbf{A}^1_0 \arrow[dl, "\pi^{12}"'] \arrow[dr, "\pi^{23}"] \arrow[dd, phantom, very near start, "\vee"] & & \\
 & \mathbf{A}^1_0 \times \mathbf{A}^1_0 \arrow[dl, "\pi^1"'] \arrow[dr, "\pi^2"] & & \mathbf{A}^1_0 \times \mathbf{A}^1_0 \arrow[dl, "\pi^1"'] \arrow[dr, "\pi^2"] & \\
\mathbf{A}^1_0 & & \mathbf{A}^1_0 & & \mathbf{A}^1_0 \\
T_{\psi^{-1}} T_\psi K_0 & & T_\psi K_0 \arrow[ll, squiggly] & & K_0 \arrow[ll, squiggly]
\end{tikzcd} \tag{13}$$

where the top square is cartesian. Now applying base change to this cartesian square, we can put all of our sheaves into one space:

$$= \mathbf{R}\pi^1_!\left(\mathbf{R}\pi^{12}_!\pi^{23*}\left(\pi^{2*}K_0 \otimes m^*\mathscr{L}_0(\psi)\right) \otimes m^*\mathscr{L}_0(\psi^{-1})\right)[2]$$

and then use the projection formula to get

$$= \mathbf{R}\pi^1_! \circ \mathbf{R}\pi^{12}_!\left(\pi^{23*}\pi^{2*}K_0 \otimes \pi^{23*}m^*\mathscr{L}_0(\psi) \otimes \pi^{12*}m^*\mathscr{L}_0(\psi^{-1})\right)[2] \qquad = \sum_y \sum_z f(z)\psi(-yz)\psi(xy)$$

Next, we want to combine $\psi(-yz)\psi(xy) = \psi(-(yz - yx)) = \psi(-y(z-x))$. Let $\alpha\colon \mathbf{A}^3 \to \mathbf{A}^2$ be defined by $(x, y, z) \mapsto (y, z - x)$; then, we obtain

$$= \mathbf{R}\pi^1_! \circ \mathbf{R}\pi^{12}_!\left(\pi^{23*}\pi^{2*}K_0 \otimes \alpha^*m^*\mathscr{L}_0(\psi)\right)[2] \qquad = \sum_y \sum_z f(z)\psi(-y(z-x))$$

We now want to change the order of summation on the function side. We note $\mathbf{R}\pi^1_! \circ \mathbf{R}\pi^{12}_! = \mathbf{R}\pi^1_! \circ \mathbf{R}\pi^{13}_!$ and $\pi^{23*}\pi^{2*} = \pi^{13*}\pi^{2*}$ to obtain

$$= \mathbf{R}\pi^1_! \circ \mathbf{R}\pi^{13}_!\left(\pi^{13*}\pi^{2*}K_0 \otimes \alpha^*m^*\mathscr{L}_0(\psi)\right)[2] \qquad = \sum_z \sum_y f(z)\psi(-y(z-x))$$

By the projection formula, we can pull out $f(z)$ on the function side:

$$= \mathbf{R}\pi^1_!\left(\pi^{2*}K_0 \otimes \mathbf{R}\pi^{13}_!\alpha^*m^*\mathscr{L}_0(\psi)\right)[2] \qquad = \sum_z f(z) \sum_y \psi(-y(z-x))$$

Now consider the diagram

$$\begin{tikzcd}
(x,y,z) \arrow[r, mapsto] & (y,z-x) \\
\mathbf{A}_0^3 \arrow[r, "\alpha"] \arrow[d, "\pi^{13}"'] & \mathbf{A}^2 \arrow[d, "\pi^2"] \\
\mathbf{A}_0^2 \arrow[r, "\beta"] & \mathbf{A}_0^1 \\
(x,z) \arrow[r, mapsto] & z-x
\end{tikzcd}$$

By base change with respect to this cartesian square, we obtain

$$= \mathbf{R}\pi^1_!\big(\pi^{2*}K_0 \otimes \beta^* \underbrace{\mathbf{R}\pi^2_!(m^*\mathscr{L}_0(\psi))}_{T_\psi\overline{\mathbf{Q}}_\ell[-1]}\big)[2]$$

which we can interpret on the functions side as thinking of $z-x$ as the fixed quantity in the inner sum, instead of z. By Lemma 7.17, we get the sheaf equivalent of (12) for functions:

$$= \mathbf{R}\pi^1_!\left(\pi^{2*}K_0 \otimes \beta^*\delta_0(-1)[-2])\right)[2] = \mathbf{R}\pi^1_!\left(\pi^{2*}K_0 \otimes \beta^*\delta_0(-1)\right) \qquad = \sum_z f(z)\cdot \begin{Bmatrix} q^n \text{ if } z-x=0 \\ 0 \quad \text{else} \end{Bmatrix}$$

Now applying base change with the cartesian square

$$\begin{tikzcd}
\mathbf{A}_0^1 \arrow[r] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, "i_0"] \\
\mathbf{A}_0^2 \arrow[r, "\beta"] & \mathbf{A}_0^1
\end{tikzcd}$$

we obtain

$$= \mathbf{R}\pi^1_!\left(\pi^{2*}K_0 \otimes \mathbf{R}\Delta_!\overline{\mathbf{Q}}_\ell(-1)\right) \qquad = \sum_z f(z)\sum_{x=z} q^n$$

By using the projection formula again, we can pull out the factor of q^n on the function side to get:

$$= \mathbf{R}\pi^1_! \circ \mathbf{R}\Delta_!\big(\Delta^*\pi^{2*}K_0 \otimes \overline{\mathbf{Q}}_\ell\big)(-1) \qquad = q^n \sum_{\substack{z\in\mathbf{F}_{q^n} \\ z=x}} f(z)$$

Finally, by the commutativity of the diagram

$$\begin{tikzcd}
 & & \mathbf{A}_0^1 \\
\mathbf{A}_0^1 \arrow[r, "\Delta"] \arrow[urr, bend left, "\mathrm{id}"] \arrow[drr, bend right, "\mathrm{id}"'] & \mathbf{A}_0^2 \arrow[ur, "\pi^1"] \arrow[dr, "\pi^2"'] & \\
 & & \mathbf{A}_0^1
\end{tikzcd}$$

we have $\mathbf{R}\pi^1_! \circ \mathbf{R}\Delta_! = \mathbf{R}\,\mathrm{id}_! = \mathrm{id}$ and $\Delta^*\pi^{2*} = \mathrm{id}^*$, i.e., they have no affect on sheaves, and so we obtain our desired result:

$$= K_0(-1) \qquad = q^n f(x). \qquad \square$$

Even though the proof for sheaves was a bit long, notice that the proof on the function side tells you how you should proceed in the proof.

Finally, we can also define an analogue of the Fourier transform in higher dimensions, and we get similar results as before.

Definition 7.21 (Partial Fourier Transform)**.** Fix our ground field $\mathbf{F}_{q^n}$. Given the "dot product" map

$$\begin{aligned} \mathbf{A}_0^r \times \mathbf{A}_0^r &\xrightarrow{m} \mathbf{A}_0^1 \\ (x, y) &\longmapsto \sum x_i y_i \end{aligned}$$

write

$$\begin{tikzcd}
& \mathbf{A}_0^r \times \mathbf{A}_0^r \times \mathbf{A}_0^{n-r} \times Y_0 \arrow[r, "m"] \arrow[dl, "\pi^1"'] \arrow[dr, "\pi^2"] & \mathbf{A}_0^1 \quad \mathscr{L}_0(\psi) \\
\mathbf{A}_0^n \times Y_0 & & \mathbf{A}_0^n \times Y_0 \\
& & K_0
\end{tikzcd}$$

Consider the sheaf $\mathscr{L}_0(\psi)$ defined on the codomain of m, and K_0 defined on the codomain of π^2. Then, we define the *partial (functor) Fourier transform* as

$$T_{\psi,r}K_0 = \mathbf{R}\pi^1_!(\pi^{2*}K_0 \otimes m^*\mathscr{L}_0(\psi))[r]. \tag{14}$$

Remark 7.22. Fix middle perversity. We would like all operations in (14) to perserve perversity, and explain the shift $[r]$:

1. The functor π^{2*} shifts perversity, so in order to preserve middle perversity we need to apply the shift $[r]$ by the codimension.
2. The functor $\mathbf{R}\pi^1_!$ is in general only left-t-exact. But Fourier inversion below will tell us that $\mathbf{R}\pi^1_!$ is right-t-exact as well.

Theorem 7.23 (Fourier Inversion)**.** *$T_{\psi^{-1},r}T_{\psi,r}K_0 = K_0(-r)$.*

Proof. Use base change and the projection formula to show

$$T_{\psi^{-1},r} \circ T_{\psi,r} = T_{\psi^{-1},r-1} \circ T_{\psi,r-1} \circ T_{\psi^{-1},r} \circ T_{\psi,r},$$

and proceed by induction on r, where the base case is Theorem 7.20. ☐

Theorem 7.24. *$T_\psi \colon \mathsf{Perv}(\mathbf{A}_0^n \times Y_0, \overline{\mathbf{Q}}_\ell) \to \mathsf{Perv}(\mathbf{A}_0^n \times Y_0, \overline{\mathbf{Q}}_\ell)$ is an equivalence.*

Before giving all of the details, we give the idea for the proof. First, just by definition of T_ψ, we have that if $K_0 \in \mathsf{Perv}(\mathbf{A}_0^n \times Y, \overline{\mathbf{Q}}_\ell)$, then $T_\psi K_0 \in {}^p\mathsf{D}^{\geq 0}(\mathbf{A}_0^n \times Y_0, \overline{\mathbf{Q}}_\ell)$. We then want to apply Fourier inversion (Theorem 7.23) to show ${}^p\tau_{\geq 1}T_\psi K_0 = 0$.

Proof. By definition, we have

$$T_\psi(K_0) = \underbrace{\mathbf{R}\pi^1_!}_{(1)}\big(\underbrace{\pi^{2*}K_0[r]}_{(2)}\underbrace{\otimes\, \pi^*\mathscr{L}_0(\psi)}_{(3)}\big).$$

Note that, for the perverse t-structure (and not for the standard constructible t-structure),

1. $\mathbf{R}\pi^1_!$ is left t-exact since π^1 is affine;
2. $K_0 \mapsto \pi^{2*}K_0[r]$ is t-exact since π^2 is equidimensional and smooth of fibre dimension r;
3. $- \otimes \pi^*\mathscr{L}_0(\psi)$ is t-exact since $\pi^*\mathscr{L}_0(\psi)$ is a local system.

Thus, $T_\psi K_0 \in {}^p\mathsf{D}^{\geq 0}(\mathbf{A}_0^n \times Y_0, \overline{\mathbf{Q}}_\ell)$.

Now to show that $T_\psi K_0$ is actually perverse, we want to show that ${}^p\tau_{\geq 1}T_\psi(K_0) = 0$. To do so, we just need to write down the correct distinguished triangles.

Consider the distinguished triangle

$$\begin{tikzcd}
{}^pH^0T_\psi B \arrow[r] \arrow[d, equal] & T_\psi B \arrow[r] & {}^p\tau_{\geq 1}T_\psi B \arrow[r, rightsquigarrow] & {} \\
{}^p\tau_{\leq 0}T_\psi B
\end{tikzcd} \tag{15}$$

Since $T_{\psi^{-1}}$ is exact, we can apply Fourier inversion (Theorem 7.23) to obtain a new distinguished triangle:

$$\begin{tikzcd}
T_{\psi^{-1}}\,{}^pH^0T_\psi B \arrow[r] & B(-r) \arrow[r, "0"] & T_{\psi^{-1}}\,{}^p\tau_{\geq 1}T_\psi B \arrow[r, rightsquigarrow] & {}
\end{tikzcd}$$

where the map $B(-r) \to T_{\psi^{-1}}\, {}^p\tau_{\geq 1} T_\psi B$ must be the zero map since $B(-r)$ is perverse. This implies that the map $T_\psi B \to {}^p\tau_{\geq 1} T_\psi B$ in (15) is also the zero map.

The rotation of (15) is the distinguished triangle

$$\begin{tikzcd}
T_\psi B \arrow[r, "0"] & {}^p\tau_{\geq 1} T_\psi B \arrow[r] & {}^pH^0 T_\psi B[1] \arrow[r, squiggly] \arrow[d, phantom, "\wr\|"] & {} \\
 & & \mathrm{Cone}\Big(T_\psi(B) \xrightarrow{0} {}^p\tau_{\geq 1} T_\psi B\Big) \arrow[d, phantom, "\|"] & \\
 & & T_\psi B[1] \oplus {}^p\tau_{\geq 1} T_\psi B &
\end{tikzcd}$$

Thus,

$${}^pH^0 T_\psi B \cong T_\psi B \oplus {}^p\tau_{\geq 1} T_\psi B[-1].$$

Applying ${}^p\tau_{\geq 1}$ to both sides of this isomorphism gives

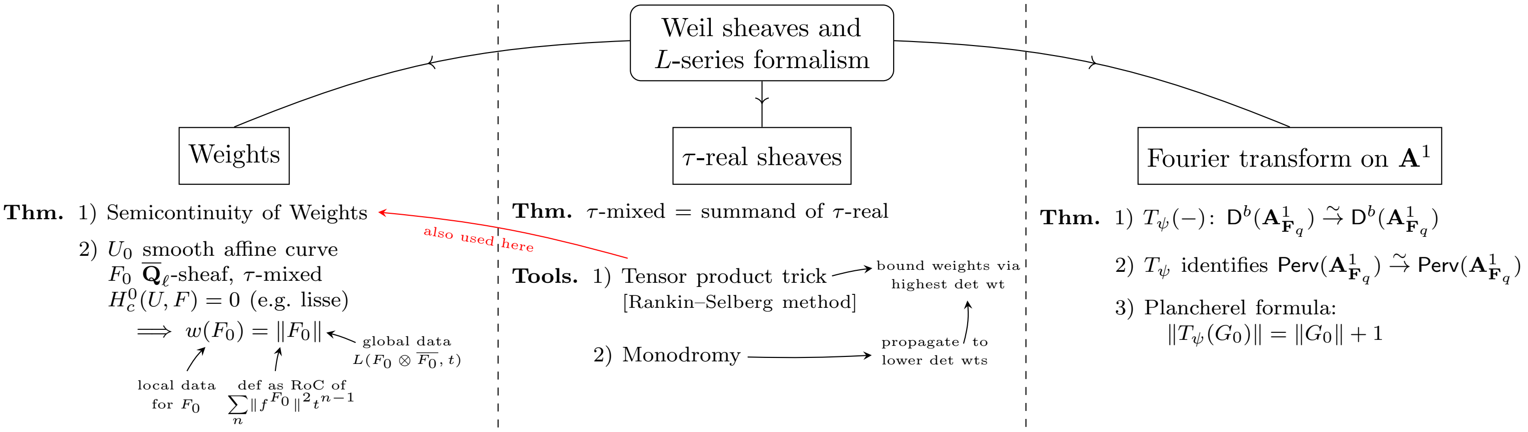
$${}^p\tau_{\geq 1}\, {}^pH^0 T_\psi B \cong {}^p\tau_{\geq 1} T_\psi B \oplus {}^p\tau_{\geq 1}\, {}^p\tau_{\geq 1} T_\psi B[-1],$$

but the left-hand side is zero, and so we have that the right-hand side is also zero. In particular, ${}^p\tau_{\geq 1} T_\psi B = 0$, and so $T_\psi B \in \mathsf{Perv}(\mathbf{A}_0^n \times Y_0, \overline{\mathbf{Q}}_\ell)$.

Finally, $T_\psi \colon \mathsf{Perv}(\mathbf{A}_0^n \times Y_0) \to \mathsf{Perv}(\mathbf{A}_0^n \times Y_0)$ is an equivalence since it has an inverse $T_{\psi^{-1}}(r)$. $\square$

8 June 6—Weil Conjectures I (Bhargav Bhatt)

We start with a *rough* chart of the logical structure of the proof of the Weil conjectures so far:



Thm. 1) Semicontinuity of Weights

2) U_0 smooth affine curve
F_0 $\overline{\mathbf{Q}}_\ell$-sheaf, τ-mixed
$H^0_c(U, F) = 0$ (e.g. lisse)
$\implies w(F_0) = \|F_0\|$

Thm. τ-mixed = summand of τ-real

Tools. 1) Tensor product trick [Rankin–Selberg method]

2) Monodromy

Thm. 1) $T_\psi(-)\colon \mathsf{D}^b(\mathbf{A}^1_{\mathbf{F}_q}) \xrightarrow{\sim} \mathsf{D}^b(\mathbf{A}^1_{\mathbf{F}_q})$

2) T_ψ identifies $\mathsf{Perv}(\mathbf{A}^1_{\mathbf{F}_q}) \xrightarrow{\sim} \mathsf{Perv}(\mathbf{A}^1_{\mathbf{F}_q})$

3) Plancherel formula:
$\|T_\psi(G_0)\| = \|G_0\| + 1$

Today we will use these results to prove the Weil conjectures.

8.1 Curve case

Deligne's theorem for curves is the following:

Theorem 8.1. *Let U_0 be a smooth affine curve over $\mathbf{F}_q$, and let F_0 be a lisse $\overline{\mathbf{Q}}_\ell$ sheaf on U_0, which is τ-mixed of weight $\leq w$. Then, $H^i_c(U, F)$ is τ-mixed of weight $\leq w + i$.*

We start with a series of reductions to reduce to the case when $U_0 = \mathbf{A}^1_0$.

1. We may assume $i = 1$: $i = 0$ vanishes and the $i = 2$ case is obvious by Poincaré duality

$$H^2_c(U, F) \cong H^0(U, F^\vee)^\vee(-1).$$

2. We may shrink U_0: If $j_0 \colon V_0 \hookrightarrow U_0$ is a dense open immersion, then the map $H^1_c(V, F|_V) \twoheadrightarrow H^1_c(U, F)$, obtained by covariant functoriality with respect to immersions, is surjective since the cokernel of $j_{0!}(F_0|_{V_0}) \to F_0$ has finite support.

3. We may extend the base field by finite extensions freely.
4. We may assume $U_0 \subset \mathbf{A}^1_{\mathbf{F}_q}$: By noether normalization, you can find a diagram that looks like

   $$\begin{tikzcd} V_0 \arrow[r, phantom, "\subset"] \arrow[d, "\pi", "\text{finite étale}"'] & U_0 \arrow[d, "\pi"', "\text{finite}"] \\ W_0 \arrow[r] & \mathbf{A}^1_{\mathbf{F}_q} \end{tikzcd}$$

   and we can replace (U_0, F_0) by $(W_0, \pi_*(F_0|_{V_0}))$, since $H^1_c(V, F|_V) \cong H^1_c(W, \pi_*(F|_V))$. Note that we cannot control the rank of $\pi_*(F_0|_{V_0})$ in this process.
5. We may assume F_0 is (geometrically) irreducible: $H^1_c(U, -)$ is a left-exact functor on local systems over U, and so we can split up F_0 into its irreducible constituents, which become geometrically irreducible after a finite base extension.
6. We may assume that F_0 is unramified at infinity, that is, that F_0 extends to a local system on $U_0 \cup \{\infty\} \subset \mathbf{P}^1_{\mathbf{F}_q}$: Pick an unramified point u in $U_0 \subset \mathbf{A}^1_{\mathbf{F}_q}$, shrink U_0 so $u \notin U_0$, and move u to ∞ by a Möbius transformation (after possibly extending the base field so that the transformation is defined over it).
7. We may assume F_0 is not (geometrically) constant, that is, not constant after passing to the algebraic closure $\overline{\mathbf{F}}_q$ of our base field, by proving this case separately. Suppose $F_0 \cong \underline{\mathbf{Q}_\ell}$, and write $j_0 \colon U_0 \hookrightarrow \mathbf{P}^1_{\mathbf{F}_q}$. Let $i_0 \colon Z_0 \hookrightarrow \mathbf{P}^1_{\mathbf{F}_q}$ be the complement of U_0. Then, we get the short exact sequence

   $$0 \longrightarrow j_{0!}\underline{\mathbf{Q}_\ell} \longrightarrow j_*\underline{\mathbf{Q}_\ell} \longrightarrow Q \longrightarrow 0.$$

   Note $j_*\underline{\mathbf{Q}_\ell} \cong \overline{\mathbf{Q}}_\ell$ since $(\mathbf{P}^1)^{\mathrm{sh}}_x - \{x\}$ is connected, and that $Q = i_*\underline{\mathbf{Q}_\ell}$. Now since the short exact sequence is completely Galois invariant, we have that the associated long exact sequence on cohomology

   $$0 \longrightarrow \mathbf{Q}_\ell^{\oplus(\#Z_0 - 1)} \longrightarrow H^1_c(U, \overline{\mathbf{Q}}_\ell) \longrightarrow H^1(\mathbf{P}^1, \overline{\mathbf{Q}}_\ell) = 0$$

   is exact, where the -1 in the exponent of $\mathbf{Q}_\ell$ comes from the fact that we get a copy of $\mathbf{Q}_\ell$ from H^0 on $\mathbf{P}^1$. Since the middle group has weight 0, we are done.

   We can also prove this using semicontinuity of weights: without knowing what Q is in the short exact sequence above, we still know Q has finite support, and that $w(Q) \leq w$.

We now state the key assertions that we will use to prove Theorem 8.1:

Key Assertions 8.2. Denote $G_0 = j_{0!}(F_0)$ and denote $j_0 \colon U_0 \hookrightarrow \mathbf{A}^1_{\mathbf{F}_q}$. Let $\psi \colon \mathbf{F}_q \to \overline{\mathbf{Q}}_\ell$ be a fixed non-trivial additive character. Then,

- (*a*) $T_\psi(G_0)$ is a sheaf, that is, the complex $T_\psi(G_0)$ is conctentrated in degree 0;
- (*b*) $H^0_c(\mathbf{A}^1, T_\psi(G_0)) = 0$;
- (*c*) $T_\psi(G_0)$ is τ-mixed.

Remark 8.3. Condition (*a*) fails if G_0 is geometrically constant, which explains why we needed reduction (7).

Proof of Theorem 8.1 assuming Key Assertions. Recall that the Fourier transform switches taking stalks and computing cohomology (Theorem 7.13), that is,

$$(T_\psi(G_0))|_{\{0\}} = \mathbf{R}\Gamma_c(\mathbf{A}^1, G)[1].$$

But the right-hand side is $\mathbf{R}\Gamma_c(\mathbf{A}^1, G)[1] = \mathbf{R}\Gamma_c(U, F)[1] = H^1_c(U, F)[0]$ by assertion (*a*).

To understand the Frobenius eigenvalues of $H^1_c(U, F)$, then, it suffices to understand the Frobenius eigenvalues, hence the weights, of the Fourier transform; in particular, we want to show

$$w(T_\psi(G_0)) \leq w + 1. \tag{16}$$

We showed (Theorem 4.11) that the weights of $T_\psi(G_0)$ satisfy $w(T_\psi(G_0)) = \|T_\psi(G_0)\|$, where we note that in order to apply Theorem 4.11, we needed assertions (*b*) and (*c*); the same Theorem also says $w(G_0) = \|G_0\|$. Therefore, (16) holds if and only if $\|T_\psi(G_0)\| \leq \|G_0\| + 1$, which is exactly the Plancherel formula (Theorem 7.15; see also Remark 7.16) from last time. ☐

Proof of Key Assertions (a) *and* (b). Instead of following [KW01, §I.6], we use the perverse equivalence (Theorem 7.24) from last time. Since (a) and (b) are both geometric statements, we can pass to the algebraic closure, and consider the sheaves F and $G = j_!(F)$, where F is an irreducible local system on U. Then, the shift $G[1]$ of G is a simple, irreducible perverse sheaf on $\mathbf{A}^1$, since j is affine. By Theorem 7.24, this is equivalent to saying $T_\psi(G[1])$ is an irreducible perverse sheaf on $\mathbf{A}^1$. But there is a classification of these:

$$T_\psi(G[1]) = \begin{cases} k_!(M[1]) & \text{where } k\colon V \hookrightarrow \mathbf{A}^1 \text{ is a dense open, and } M \text{ is an irreducible local system on } V\text{; or} \\ i_*N & \text{where } i\colon \{x\} \hookrightarrow \mathbf{A}^1 \text{ and } N \text{ is an irreducible local system on } \{x\}\text{, i.e., } N = \underline{\mathbf{Q}_\ell}. \end{cases}$$

We want to show that the second situation doesn't happen. Note that by the way the perverse t-structure works, we have a shift [1] in the first case but not the second.

So suppose for sake of contradiction that $T_\psi(G[1]) = i_*\underline{\mathbf{Q}_\ell}$. We know that $T_\psi(\mathscr{L}(\psi_{-x})) = i_*\underline{\mathbf{Q}_\ell}[-1]$, and so G is isomorphic to the Artin–Schreier sheaf $L(\psi_{-x})$. But these are always ramified at infinity (since otherwise you get a finite étale cover of $\mathbf{P}^1$), unless $x = 0$, in which case we have that G is isomorphic to the constant sheaf $\underline{\mathbf{Q}_\ell}$, which we've already ruled out.

Thus, $T_\psi(G[1]) = k_!(M[1])$ where $k\colon V \hookrightarrow \mathbf{A}^1$ is a dense open, and M is an irreducible local system on V. Canceling the shifts on either side, we have that $T_\psi(G) = k_!(M)$, in which case the assertions are clear: this is clearly a sheaf, and $H^0_c(\mathbf{A}^1, T_\psi(G)) = H^0_c(V, M) = 0$. ☐

We therefore see that (a) and (b) are really just consequence of the abstract Fourier transform machinery, without having to do anything new and difficult. We now move onto showing the remaining Key Assertion:

Proof of Key Assertion (c). We want to use the last piece of machinery we have yet to use: the fact that summands of τ-real sheaves are τ-mixed. Note that this is really the only way we know of to prove that a sheaf is τ-mixed.

Consider the Fourier transform setup from last time

$$\begin{tikzcd}
 & \mathbf{A}^1_{\mathbf{F}_q} \times \mathbf{A}^1_{\mathbf{F}_q} \arrow[r, "m"] \arrow[dl, "\pi^1"'] \arrow[dr, "\pi^2"] & \mathbf{A}^1_{\mathbf{F}_q} \\
\mathbf{A}^1_{\mathbf{F}_q} & & \mathbf{A}^1_{\mathbf{F}_q}
\end{tikzcd}$$

Recall that $G_0 = j_{0!}(F_0)$, and so its Fourier transform is $T_\psi(G_0) \coloneqq \mathbf{R}\pi^1_!(\pi^{2*}G_0 \otimes m^*\mathscr{L}(\psi))[1]$.

Consider the following τ-real sheaf formed by summing the sheaf used in the definition of the Fourier transform (before taking $\mathbf{R}\pi^1_!$) with its dual:

$$H_0 = \underbrace{\pi^{2*}(j_{0!}(F_0)) \otimes m^*(\mathscr{L}(\psi))}_{A} \oplus \underbrace{(\pi^{2*}(j_{0!}(F_0^\vee)) \otimes m^*(\mathscr{L}(\psi^{-1})))}_{B} \otimes \mathscr{L}_{\mathfrak{b}},$$

where $\mathfrak{b} \in \overline{\mathbf{Q}}_\ell$ is such that $\tau(\mathfrak{b}) = q^w$. By construction, H_0 is τ-real. Now $R^i\pi^1_!(A) = 0$ if $i \neq 1$ by assertion (a). In the same way, you can show $R^i\pi^1_!(B) = 0$ if $i \neq 1$. By the Grothendieck trace formula for $\mathbf{R}\pi^1_!(H_0)_{\overline{x}}$, where $\overline{x}$ is a geometric point living over a closed point $x \in |\mathbf{A}^1|$, since the higher direct images for $i \neq 1$ vanish, we have

$$\det\Bigl(1 - t \cdot F_x^{d(x)} \Bigm| \bigl(R^1\pi^1_!(H_0)\bigr)_{\overline{x}}\Bigr) = \prod_{y \in |(\pi^1)^{-1}(x)|} \det\Bigl(1 - t \cdot F_y^{d(y)} \Bigm| (H_0)_{\overline{y}}\Bigr)^{-1}$$

The right-hand side has $\mathbf{R}$-coefficients by construction, and so the left-hand side has $\mathbf{R}$-coefficients, that is, $\mathbf{R}\pi^1_!(H_0)$ is τ-real. By Theorem 6.9, the summand $\mathbf{R}\pi^1_!(A) = T_\psi(G_0)[-1]$ is τ-mixed. ☐

This concludes the proof of the Weil conjectures in the curve case.

8.2 General Case

Theorem 8.4. *Let X_0 be a scheme of finite type over $\mathbf{F}_q$, and let F_0 be a $\overline{\mathbf{Q}}_\ell$ sheaf on X_0, which is τ-mixed of weight $\leq w$. Then, $H^i_c(X,F)$ is τ-mixed of weight $\leq w+i$.*

Corollary 8.5. *If $f_0\colon X_0 \to Y_0$ is a smooth, proper map of schemes of finite type, and F_0 is τ-pure of weight β, then $R^i f_{0!}(F_0)$ is τ-pure of weight $\beta + i$.*

Proof of Corollary 8.5. Apply Theorem 8.4 and base change to identify stalks. $\square$

Proof of Theorem 8.4. We induce on the dimension d of X_0.

1. $d=0$ is okay: it's just a finite union of points.
2. We may replace (X_0, F_0) by $(U_0, F_0|_{U_0})$ just as before, where $U_0 \overset{j_0}{\hookrightarrow} X_0$ is a dense open: If

$$U_0 \overset{j_0}{\hookrightarrow} X_0 \overset{i_0}{\hookleftarrow} Z_0 = X_0 \setminus U_0,$$

   then we get the short exact sequence

$$0 \longrightarrow j_{0!}(F_0|_{U_0}) \longrightarrow F_0 \longrightarrow i_*(F_0|_{Z_0}) \longrightarrow 0$$

   and the long exact sequence on cohomology says

$$\cdots \longrightarrow H^i_c(U, F|_U) \longrightarrow H^i_c(X,F) \longrightarrow H^i_c(Z, F|_Z) \longrightarrow \cdots$$

   is exact. We can control the $H^i_c(Z, F|_Z)$ term by induction.
3. We may extend base fields.
4. To finish the proof, first reduce to the case where X_0 is a smooth affine variety, and F_0 is a local system on X_0 by (2). Then, we may assume there exists a map $\pi\colon X_0 \to Y_0$ is a smooth, affine map of relative dimension 1, e.g., by using noether normalization to reduce to the case where X_0 is an (open subset of an) affine space, and then by using a coordinate projection, passing to smaller open subsets as necessary to keep things smooth. We are therefore in a situation where we have X_0 fibred over Y_0 by curves.

We then have a Leray spectral sequence

$$E_2^{i,j}\colon H^i_c(Y, R^j\pi_!(F)) \Rightarrow H^{i+j}_c(X,F).$$

Claim 8.6. *$R^j\pi_!(F_0)$ is τ-mixed of weight $\leq w+j$.*

The Claim plus induction on dimension implies $E_2^{i,j}$ is τ-mixed of weights $\leq w+i+j$, and so $E_\infty^{i,j}$ is τ-mixed of weights $\leq w+i+j$. There is a subtlety in showing this Claim, however, and requires one more reduction, which we will incorporate in the proof next time. $\square$

Next time we will finish the end of the proof of Theorem 8.4 and discuss Hard Lefschetz.

9 June 8—Weil Conjectures II & Hard Lefschetz (Bhargav Bhatt)

9.1 Weil Conjectures

Last time, we proved:

Theorem 9.1. *Let U_0 be a smooth affine curve over $\mathbf{F}_q$, and let F_0 be a smooth $\overline{\mathbf{Q}}_\ell$-sheaf, which is τ-mixed of weights $\leq w$. Then, $H^i_c(U,F)$ is τ-mixed of weights $\leq w+i$.*

Example 9.2. Let E_0 be an elliptic curve over $\mathbf{F}_q$, and choose two rational points $x, y \in E_0(\mathbf{F}_q)$ that are distinct. Let U_0 be the complement $E_0 \setminus \{x,y\}$. This is a smooth affine curve. Let $F_0 = \overline{\mathbf{Q}}_\ell$.

Claim 9.3. *$H^1_c(U, \overline{\mathbf{Q}}_\ell)$ is τ-mixed of weights $0, 1$, i.e., some constituents have weight 0 and others have weight 1.*

Proof. Let $U_0 \overset{j_0}{\hookrightarrow} E_0 \overset{i_0}{\hookleftarrow} Z_0 = \{x, y\}$. Then, we have the usual short exact sequence

$$0 \longrightarrow j_{0!}(\underline{\mathbf{Q}_\ell}) \longrightarrow \underline{\mathbf{Q}_\ell} \longrightarrow i_{0*}(\underline{\mathbf{Q}_\ell}) \longrightarrow 0$$

Then, the long exact sequence on cohomology says

$$\begin{tikzcd} 0 \arrow[r] & \dfrac{H^0(Z, \overline{\mathbf{Q}}_\ell)}{H^0(E, \overline{\mathbf{Q}}_\ell)} \arrow[r] \arrow[d, phantom, "\wr\|"] & H^1_c(U, \overline{\mathbf{Q}}_\ell) \arrow[r] & H^1(E, \underline{\mathbf{Q}_\ell}) \arrow[r] \arrow[d, phantom, "\|"] & 0 \\ & \overline{\mathbf{Q}}_\ell(0) & & \text{weight 1 rep} \\ & & & \text{of } G_{\mathbf{F}_q} \end{tikzcd}$$

□

Remarks 9.4.

1. If F_0 is τ-pure of weight w (we reduced to this case anyways), then $H^1_c(U, F)$ is τ-mixed of weights $\leq w + 1$, and $H^2_c(U, F)$ is τ-pure of weight $w + 2$ (by Poincaré duality, since H^0 is a stalk). We used this to show there was no cancellation in the L-function
$$L(U_0, F_0, t) = \frac{\det\bigl(1 - t \cdot F \mid H^1_c(U, F)\bigr)}{\det(1 - t \cdot F \mid H^2_c(U, F))}$$
and so if $L(U_0, F_0, t)$ has $\mathbf{R}$-coefficients, the same also holds for $\det\bigl(1 - t \cdot F \bigm| H^i_c(U, F)\bigr)$ for both $i = 1, 2$.
2. Fix $f_0 \colon X_0 \to Y_0$ a smooth affine map of relative dimension 1, and let F_0 be a τ-pure smooth $\overline{\mathbf{Q}}_\ell$-sheaf on X_0 of weight w. We then want to understand the higher pushforwards of F_0: $R^i f_{0!}(F_0)$ is τ-mixed of weight $\leq w + i$, and is τ-pure of weight $w + 2$ if $i = 2$.

We now prove the general case.

Theorem 9.5. *Let X_0 be a scheme of finite type over $\mathbf{F}_q$, and let F_0 be any τ-mixed sheaf of weight $\leq w$. Then, $H^i_c(X, F)$ is τ-mixed of weight $\leq w + i$.*

Corollary 9.6. *If X_0 is a smooth proper variety over $\mathbf{F}_q$ of dimension d, and F_0 is τ-pure of weight w, then $H^i(X, F)$ is also τ-pure of weight $w + i$.*

Proof of Corollary 9.6. By the Theorem 9.5, we have that $H^i(X, F)$ is τ-mixed of weights $\leq w + i$. By duality,
$$H^i(X, F) \cong (H^{2d-i}(X, F^\vee))^\vee(-d).$$
$H^{2d-i}(X, F^\vee)$ has weights $\leq -w + 2d - i$, and so the right-hand side has weights $\geq w - 2d + i + 2d = w + i$. □

Proof of Theorem 9.5. Let $d = \dim(X_0)$. We work by induction on d. We make the following reductions, as we did last time:

- We may assume X_0 is a smooth affine variety, and F_0 is a smooth $\overline{\mathbf{Q}}_\ell$-sheaf.
- We may assume F_0 is actually τ-pure of weight w (we forgot to say this last time), by using long exact sequences.
- We may assume that there exists $\pi_0 \colon X_0 \to Y_0$ a smooth affine map of relative dimension 1.

[It might be best to do the reductions in the opposite order.]

We now have the following Leray spectral sequence:

$$E_2^{i,j} \colon H^i_c(Y, R^j \pi_!(F)) \Rightarrow H^{i+j}_c(X, F)$$

Remark 9.4(2) to Theorem 9.1 shows that $R^j \pi_!(F)$ is τ-mixed of weight $\leq w + j$. Induction then implies the result: $E_2^{i,j}$ is τ-mixed of weights $\leq w + i + j$, and so $E_\infty^{i,j}$ is τ-mixed of weights $\leq w + j + i$. □

Remark 9.7. Using similar arguments, one can show: If $f_0 \colon X_0 \to Y_0$ is a (separated) map of finite type $\mathbf{F}_q$-schemes, and F_0 is a τ-mixed sheaf of weights $\leq w$ on X_0, then $R^i f_{0!}(F_0)$ is τ-mixed of weights $\leq w + i$.

There is also a dual version to Theorem 9.5:

Theorem 9.5′. *Let X_0 be a smooth variety over $\mathbf{F}_q$, and let F_0 be τ-pure of weight w and lisse. Then, $H^i(X, F)$ is τ-mixed of weights $\geq w + i$.*

Proof. $H^i(X, F) \cong H^{2d-i}_c(X, F^\vee)^\vee(-d)$, and conclude as in the proof of Corollary 9.6. □

9.2 Hard Lefschetz

We now want to prove Hard Lefschetz, which is a statement about how capping with the Chern class of an ample line bundle acts on cohomology. We first start with a theorem about monodromy representations:

Theorem 9.8. *Let X be smooth and geometrically connected over $\mathbf{F}_q$. Let F_0 be τ-pure of weight w and lisse. Recall that this corresponds to some representation $\rho \in \mathsf{Rep}_{\overline{\mathbf{Q}}_\ell}(\pi_1^{\text{arith}}(X_0))$. Then, $\rho|_{\pi_1^{\text{geom}}(X)}$ is semisimple, that is, "any pure sheaf has semisimple geometric monodromy."*

Proof. We work by induction on $l(\rho)$, the length of ρ, defined to be the number of semisimple constituents of ρ in $\mathsf{Rep}(\pi^{\text{arith}}(X_0))$. If $l(\rho) = 1$, then ρ is irreducible, and since $\pi_1^{\text{geom}}(X) \subset \pi_1^{\text{arith}}(X_0)$ is normal, Clifford's theorem implies $\rho|_{\pi_1^{\text{geom}}(X)}$ is semisimple.

Now assume $l(\rho) > 1$, and so there exists a short exact sequence

$$0 \longrightarrow A_0 \longrightarrow F_0 \longrightarrow B_0 \longrightarrow 0 \tag{17}$$

where A_0, B_0 are both τ-pure of the same weight w as F_0, and A_0, B_0 are both nonzero. We know by induction that A_0, B_0 both induce semisimple representations. We want to show that (17) is split after removing the zeros. Note that (17) is classified by some element $c \in \operatorname{Ext}^1_{X_0}(B_0, A_0)$. We want that the image of c in $\operatorname{Ext}^1_X(B, A)$ is zero. This follows by combining two observations:

1. $\operatorname{im}(c)$ is Frob-invariant of $\operatorname{Ext}^1_X(B, A) = H^1(X, B^\vee \otimes A)$.
2. $B^\vee \otimes A$ is τ-pure of weight 0, and so Theorem $2'$ says that $H^1(X, B^\vee \otimes A)$ is τ-mixed of weights ≥ 1. This creates a mismatch of weights: $\operatorname{im}(c)$ cannot simultaneously be Frobenius-invariant and have eigenvalues of weight ≥ 1. ☐

Example 9.9. Say $f \colon X \to S$ is a smooth proper map over $\overline{\mathbf{F}}_q$, and S is smooth. Then, Theorem 9.8 implies $R^i f_* \underline{\mathbf{Q}_\ell} \in \mathsf{Rep}(\pi_1(S))$ is semisimple (it is lisse since f is proper smooth, and τ-pure of weight i by the Weil conjectures and the duality argument).

This same statement is true over complex numbers, but we are unaware of which came first.

We now introduce "Hard Lefschetz" and "Not-so-hard Lefschetz."

Recall 9.10. Let k be an algebraically closed field; we will mostly consider the case when $k = \overline{\mathbf{F}}_q$. Let X be a smooth projective variety over k of dimension d, and $L \in \operatorname{Pic}(X)$ an ample line bundle. Then, we have the *first Chern class* $c_1(L) \in H^2(X, \mathbf{Q}_\ell(1))$ of L.

Theorem 9.11.

1. *[Weak Lefschetz] If $H \subset X$ is a smooth divisor in the linear system $|L|$, then the restriction map*
$$H^i(X, \mathbf{Q}_\ell) \xrightarrow{\text{Res}} H^i(H, \mathbf{Q}_\ell)$$
is bijective if $i < d - 1$, and injective if $i = d - 1$.
2. *[Hard Lefschetz] For each $0 \leq i \leq d$, the map*
$$H^{d-i}(X, \mathbf{Q}_\ell) \xrightarrow{c_1(L)^i} H^{d+i}(X, \mathbf{Q}_\ell)(i)$$
is bijective.

Corollary 9.12. *Say d is even, and let $b_i = \dim H^i(X, \mathbf{Q}_\ell)$. Then, $b_0 \leq b_2 \leq b_4 \leq \cdots \leq b_d$. Likewise, $b_1 \leq b_3 \leq b_5 \leq \cdots \leq b_{d-1}$. Also, $b_i = b_{2d-i}$. (There is a similar statement for d odd.)*

We want to show the Lefschetz theorems using the monodromy results we've proved. There is a nice theory of Lefschetz pencils, of which we will use a tiny portion.

Fact 9.13. After possibly replacing L with $L^{\otimes m}$, there exist two general elements of $|L|$ that intersect in a codimension 2 set $\Delta \subset X$. Then, we can blow up Δ to obtain a pencil $\overline{Y}$:

$$\begin{tikzcd}
Y_s \arrow[r, hook] \arrow[d] & Y \arrow[r, phantom, "\subset"] \arrow[d, "\pi|_U"] & \overline{Y} \arrow[r, "f"] \arrow[d, "\pi"] & X \\
\{s\} \arrow[r, hook] & U \arrow[r, phantom, "\underset{\text{open}}{\subset}"] & \mathbf{P}^1 &
\end{tikzcd}$$

where f is the blow up of $\Delta \subset X$, hence proper birational; π is a proper map, with all fibres of dimension $d-1$ with at worst a single ordinary double point; and $\pi|_U$ is proper and smooth of relative dimension $d-1$.

In this setup, the cohomology of X and fibres of Y are nicely related: If $s \in U$ is a geometric point, then for $i \leq d-1$,

$$H^i(X, \mathbf{Q}_\ell) \xrightarrow{\sim} \left(H^i(Y_s, \mathbf{Q}_\ell)\right)^{\pi_1(U)}.$$

This is saying that in the range where Weak Lefschetz only gives an injection, we have a bijection after passing to invariants.

Proof of Hard Lefschetz. We work by induction on the dimension d; the $d = 0$ case is stupid.

Choose a Lefschetz pencil:

$$\begin{tikzcd}
Y_s \arrow[r, hook] \arrow[d] & Y \arrow[r, hook] \arrow[d, "\pi|_U"] & \overline{Y} \arrow[r, "f"] \arrow[d, "\pi"] & X \\
\{s\} \arrow[r, hook] & U \arrow[r, hook] & \mathbf{P}^1 &
\end{tikzcd}$$

Set $h\colon Y_s \hookrightarrow X$, so $h^*\colon H^i(X) \to H^i(Y_s)$. Then, Weak Lefschetz says

$$h^* \quad \text{is an} \begin{cases} \text{isomorphism} & \text{if } i < d-1 \\ \text{injection} & \text{if } i = d-1 \end{cases}$$

Duality says

$$h_*\colon H^j(Y_s) \to H^{j+2}(X) \quad \text{is a(n)} \begin{cases} \text{isomorphism} & \text{if } j \geq d \\ \text{surjection} & \text{if } j = d-1 \end{cases}$$

Now consider

$$\begin{tikzcd}
H^{d-i}(X) \arrow[r, "c_1(L)^{i-1}"] \arrow[d, "h^*"] & H^{d+i-2}(X) \arrow[r, "c_1(L)"] \arrow[d, "h^*"] & H^{d+i(X)} \\
H^{d-i}(Y_s) \arrow[r, "c_1(L)^{i-1}"] & H^{d+i-2}(Y_s) \arrow[ur, "h_*"'] &
\end{tikzcd}$$

The diagram commutes since $h_*h^*(\alpha) = \alpha \cdot h_*h^*(1) = \alpha \cdot h_*(1) = \alpha \cdot c_1(L)$. Now if $i \geq 2$, then h^*, h_* are isomorphisms, and so by the diagram $c_1(L)^i$ is an isomorphism by induction.

Now assume $i = 1$. Then, h^* is injective, and h_* is surjective. Using Poincaré duality, we have $H^{d-1}(X) \overset{c_1(L)}{\longrightarrow} H^{d+1}(X)$ is an isomorphism, if and only if the pairing

$$\begin{aligned} H^{d-1}(X) \times H^{d-1}(X) &\longrightarrow H^{2d}(X) \cong \mathbf{Q}_\ell \\ (a, b) &\longmapsto a \cup c_1(L)b \end{aligned} \tag{18}$$

is non-degenerate. To prove this, using h^*, it suffices to show the standard pairing

$$H^{d-1}(Y_s) \times H^{d-1}(Y_s) \longrightarrow H^{2d-2}(Y_s) \cong \mathbf{Q}_\ell$$

is non-degenerate on $h^*(H^{d-1}(X))$, since we have the commutative diagram

$$\begin{tikzcd}
(a, b) \arrow[rr, mapsto] & & a \cup c_1(L)b \\
H^{d-1}(X) \times H^{d-1}(X) \arrow[rr] \arrow[d, "h^*"] & & H^{2d}(X) \arrow[d, "c_1(L)^{-1}"] \\
H^{d-1}(Y_s) \times H^{d-1}(Y_s) \arrow[rr] & & H^{2d-2}(Y_s) \\
(a, b) \arrow[rr, mapsto] & & a \cup b
\end{tikzcd}$$

We observe:

- All objects have an action of $\pi_1(U)$;
- $H^{d-1}(Y_s)^{\pi_1(U)} \cong H^{d-1}(X)$;
- Theorem 9.8 implies $H^{d-1}(Y_s) \cong H^{d-1}(Y_s)^{\pi_1(U)} \oplus Q$, where $Q^{\pi_1(U)} = 0$.

We therefore have that the pairing (18) is nondegenerate. □

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